Analytical Mechanics

Exercises 1.1-1.8

(Exercise descriptions [with possible slight modifications] from Analytical Mechanics by Fowles and Cassiday, 7th International Student Edition. Solutions by: Waves and Tensors)

Exercise 1.1: Given the two vectors $\mathbf{A} = \mathbf{i} + \mathbf{j}$ and $\mathbf{B} = \mathbf{j} + \mathbf{k}$, find the following:
(a) $\mathbf{A} + \mathbf{B}$ and $|\mathbf{A} + \mathbf{B}|$,
(b) $3\mathbf{A} - 2\mathbf{B}$,
(c) $\mathbf{A} \cdot \mathbf{B}$,
(d) $\mathbf{A} \times \mathbf{B}$ and $|\mathbf{A} \times \mathbf{B}|$.

Solution:

(a) From (1.3.3) (vector addition) we get:

$\mathbf{A} = (1, 1, 0), \mathbf{B} = (0, 1, 1)$

$\mathbf{A} + \mathbf{B} = (1, 2, 1) = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$

From (1.3.9) (magnitude of a vector) we get:

$|\mathbf{A} + \mathbf{B}| = (1^2 + 2^2 + 1^2)^{\frac{1}{2}} = \sqrt{6}$

(b) From (1.3.3) and (1.3.4) (multiplication by a scalar) we get:

$3\mathbf{A} - 2\mathbf{B} = (3, 3, 0) - (0, 2, 2) = (3, 1, -2) = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$

(c) From (1.4.1) we get:

$\mathbf{A} \cdot \mathbf{B} = 1 \cdot 0 + 1 \cdot 1 + 0 \cdot 1 = 1$

(d) From (1.5.1) we get:

$\mathbf{A} \times \mathbf{B} = (1 \cdot 1 - 0 \cdot 1, 0 \cdot 0 - 1 \cdot 1, 1 \cdot 1 - 1 \cdot 0) = (1, -1, 1) = \mathbf{i} - \mathbf{j} + \mathbf{k}$

Again from (1.3.9) we get:
$|\mathbf{A} \times \mathbf{B}| = (1^2 + (-1)^2 + 1^2)^{\frac{1}{2}} = \sqrt{3}.$

Exercise 1.2: Given the three vectors $\mathbf{A} = 2\mathbf{i} + \mathbf{j}, \mathbf{B} = \mathbf{i} + \mathbf{k}$, and $\mathbf{C} = 4\mathbf{j}$, find the following:
(a) $\mathbf{A} \cdot (\mathbf{B} + \mathbf{C})$ and $(\mathbf{A} + \mathbf{B}) \cdot \mathbf{C}$,
(b) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$ and $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$,
(c) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ and $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$.

Solution:

(a) $\mathbf{B} + \mathbf{C} = (1, 0, 1) + (0, 4, 0) = (1, 4, 1) = \mathbf{i} + 4\mathbf{j} + \mathbf{k}$

$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = 2 \cdot 1 + 1 \cdot 4 + 0 \cdot 1 = 6$

$\mathbf{A} + \mathbf{B} = (2, 1, 0) + (1, 0, 1) = (3, 1, 1) = 3\mathbf{i} + \mathbf{j} + \mathbf{k}$

$(\mathbf{A} + \mathbf{B}) \cdot \mathbf{C} = 3 \cdot 0 + 1 \cdot 4 + 1 \cdot 0 = 4$

(b) $\mathbf{B} \times \mathbf{C} = (0 \cdot 0 - 1 \cdot 4, 1 \cdot 0 - 1 \cdot 0, 1 \cdot 4 - 0 \cdot 0) = (-4, 0, 4) = -4\mathbf{i} + 4\mathbf{k}$

$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = 2 \cdot (-4) + 1 \cdot 0 + 0 \cdot 4 = -8$

$\mathbf{A} \times \mathbf{B} = (1 \cdot 1 - 0 \cdot 0, 0 \cdot 1 - 2 \cdot 1, 2 \cdot 0 - 1 \cdot 1) = (1, -2, -1) = \mathbf{i} - 2\mathbf{j} - \mathbf{k}$

$(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = 1 \cdot 0 - 2 \cdot 4 - 1 \cdot 0 = -8$

$$
\begin{aligned}
\text{(c)}\quad \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= (1 \cdot 4 - 0 \cdot 0, 0 \cdot (-4) - 2 \cdot 4, 2 \cdot 0 - 1 \cdot (-4)) \\
&= (4, -8, 4) \\
&= 4\mathbf{i} - 8\mathbf{j} + 4\mathbf{k}
\end{aligned}
$$

$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = (-2 \cdot 0 - (-1) \cdot 4, (-1) \cdot 0 - 1 \cdot 0, 1 \cdot 4 - (-2) \cdot 0 = (4, 0, 4) = 4\mathbf{i} + 4\mathbf{k}$

Exercise 1.3: Find the angle between the vectors $\mathbf{A} = a\mathbf{i} + 2a\mathbf{j}$ and $\mathbf{B} = a\mathbf{i} + 2a\mathbf{j} + 3a\mathbf{k}$ (*Note*: These two vectors define a face diagonal and a body diagonal of a rectangular block of sides $a, 2a$ and $3a$.)

Solution:

From (1.4.6) we get:

$\cos\theta = \frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{A}||\mathbf{B}|} = \frac{a^2+4a^2+0}{\sqrt{a^2+4a^2}\cdot\sqrt{a^2+4a^2+9a^2}} = \frac{5a^2}{\sqrt{5a^2}\cdot\sqrt{14a^2}} = \sqrt{\frac{5}{14}}$

$\Rightarrow \theta \approx 58,9^\circ$

Exercise 1.4: Consider a cube whose edges are each of unit length. One corner coincides with the origin of an xyz Cartesian coordinate system. Three of the cube's edges extend from the origin along the positive direction of each coordinate axis. Find the vector that begins at the origin and extends
(a) along a major diagonal of the cube,
(b) along the diagonal of the lower face of the cube.
(c) Calling these vectors $\mathbf{A}$ and $\mathbf{B}$, find $\mathbf{C} = \mathbf{A} \times \mathbf{B}$.
(d) Find the angle between $\mathbf{A}$ and $\mathbf{B}$.

Solution:

(a) Major diagonal $= \mathbf{A} = (1,1,1) = \mathbf{i} + \mathbf{j} + \mathbf{k}$

(b) $\mathbf{B} = (1,1,0) = \mathbf{i} + \mathbf{j}$

(c) $\mathbf{C} = \mathbf{A} \times \mathbf{B} = (1 \cdot 0 - 1 \cdot 1, 1 \cdot 1 - 1 \cdot 0, 1 \cdot 1 - 1 \cdot 1) = (-1,1,0) = -\mathbf{i} + \mathbf{j}$

(d) $\cos\theta = \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}||\mathbf{B}|} = \frac{1 \cdot 1 + 1 \cdot 1 + 1 \cdot 0}{\sqrt{1^2+1^2+1^2} \cdot \sqrt{1^2+1^2}} = \frac{2}{\sqrt{3} \cdot \sqrt{2}} = \sqrt{\frac{2}{3}}$

$\Rightarrow \theta \approx 35,3^\circ$

Exercise 1.5: Assume that two vectors $\mathbf{A}$ and $\mathbf{B}$ are known. Let $\mathbf{C}$ be an unknown vector such that $\mathbf{A} \cdot \mathbf{C} = u$ is a known quantity and $\mathbf{A} \times \mathbf{C} = \mathbf{B}$. Find $\mathbf{C}$ in terms of $\mathbf{A}, \mathbf{B}, u$, and the magnitude of $\mathbf{A}$.

Solution:

$$\begin{cases} \mathbf{A} \cdot \mathbf{C} = u & (1) \\ \mathbf{A} \times \mathbf{C} = \mathbf{B} & (2) \end{cases}$$

From (1.7.1) we get:

$(2) \Rightarrow \mathbf{A} \cdot (\mathbf{A} \times \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} = 0$
Also:

$(2) \Rightarrow \mathbf{C} \cdot (\mathbf{A} \times \mathbf{C}) = \mathbf{B} \cdot \mathbf{C} = 0$

Based on these we make an educated guess ($\mathbf{A} \neq 0$):

$\mathbf{C} = \frac{1}{|\mathbf{A}|^2}(u\mathbf{A} - \mathbf{A} \times \mathbf{B})$

Let us check that our guess satisfies equations (1) and (2).

(1) $\mathbf{A} \cdot \mathbf{C} = \frac{1}{|\mathbf{A}|^2}(u\mathbf{A} \cdot \mathbf{A} - \mathbf{A} \cdot (\mathbf{A} \times \mathbf{B})) = \frac{1}{|\mathbf{A}|^2} \cdot u|\mathbf{A}|^2 = u$.

(2) $\mathbf{A} \times \mathbf{C} = \frac{1}{|\mathbf{A}|^2}(u\mathbf{A} \times \mathbf{A} - \mathbf{A} \times (\mathbf{A} \times \mathbf{B})) = \frac{1}{|\mathbf{A}|^2}(-\mathbf{A}(\mathbf{A} \cdot \mathbf{B}) + \mathbf{B}(\mathbf{A} \cdot \mathbf{A})) = \frac{1}{|\mathbf{A}|^2} \cdot |\mathbf{A}|^2\mathbf{B} = \mathbf{B}$.

So the guess is a solution. For $\mathbf{A} = 0$ we get $\mathbf{B} = 0$ and $u = 0$, and $\mathbf{C}$ can be any vector in $\mathbb{R}^3$.

Exercise 1.6: Given the time-varying vector

$$\mathbf{A} = \mathbf{i}\alpha t + \mathbf{j}\beta t^2 + \mathbf{k}\gamma t^3,$$

where α, β and γ are constants, find the first and second time derivatives $\frac{d\mathbf{A}}{dt}$ and $\frac{d^2\mathbf{A}}{dt^2}$.

Solution:

From (1.10.3) we get:

$\frac{d\mathbf{A}}{dt} = \alpha\mathbf{i} + 2\beta t\mathbf{j} + 3\gamma t^2\mathbf{k}$

From (1.10.8) we get:

$\frac{d^2\mathbf{A}}{dt^2} = 0 \cdot \mathbf{i} + 2\beta\mathbf{j} + 6\gamma t\mathbf{k} = 2\beta\mathbf{j} + 6\gamma t\mathbf{k}$

Exercise 1.7: For what value (or values) of q is the vector $\mathbf{A} = q\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ perpendicular to the vector $\mathbf{B} = q\mathbf{i} - q\mathbf{j} + 2\mathbf{k}$?

Solution:

Vectors $\mathbf{A}$ and $\mathbf{B}$ are perpendicular when $\mathbf{A} \cdot \mathbf{B} = 0$:

$\mathbf{A} \cdot \mathbf{B} = q^2 - 3q + 2 = 0$

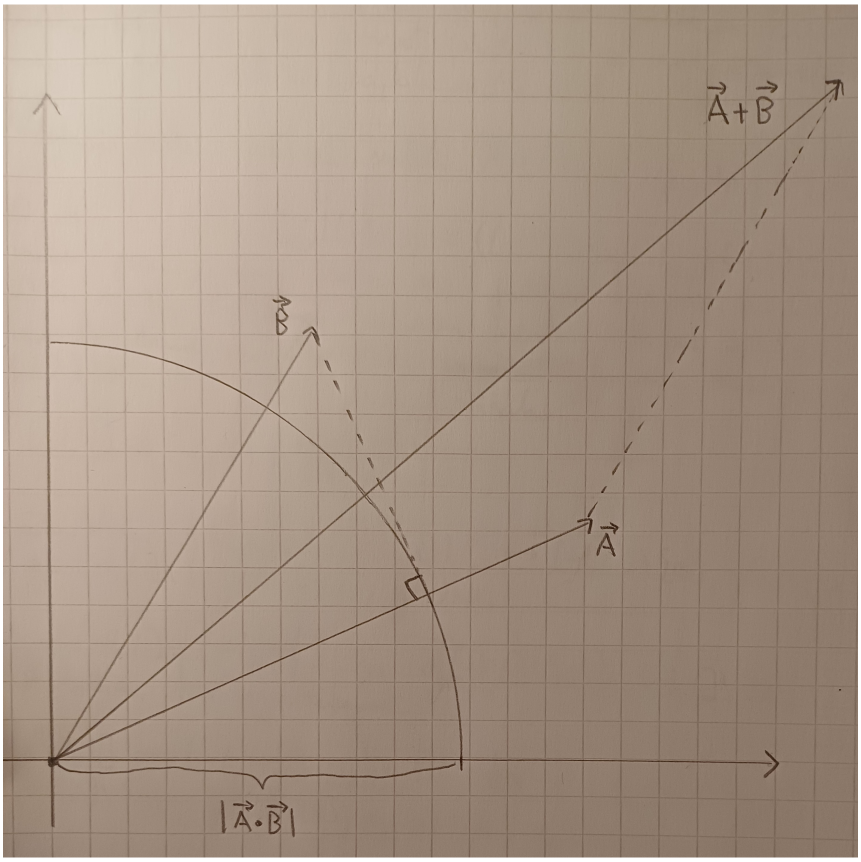
$\Rightarrow q = \frac{-(-3) \pm \sqrt{(-3)^2 - 4 \cdot 1 \cdot 2}}{2 \cdot 1} = \frac{3 \pm \sqrt{9-8}}{2} = \frac{3 \pm 1}{2}$

$\mathbf{A}$ and $\mathbf{B}$ are perpendicular when $q = 1$ or $q = 2$.

Exercise 1.8: Give an algebraic proof and a geometric proof of the following relations:

$$|\mathbf{A}+\mathbf{B}| \leq |\mathbf{A}|+|\mathbf{B}|$$
$$|\mathbf{A}\cdot\mathbf{B}| \leq |\mathbf{A}||\mathbf{B}|.$$

Solution:



From the picture we see that the circle's radius is $|\mathbf{A}\cdot\mathbf{B}|$ and clearly $\sqrt{|\mathbf{A}\cdot\mathbf{B}|} \leq |\mathbf{A}\cdot\mathbf{B}| \leq |\mathbf{A}|$ and $\sqrt{|\mathbf{A}\cdot\mathbf{B}|} \leq |\mathbf{A}\cdot\mathbf{B}| \leq |\mathbf{B}|$ so multiplied together we get:

$\sqrt{|\mathbf{A}\cdot\mathbf{B}|}\sqrt{|\mathbf{A}\cdot\mathbf{B}|} = |\mathbf{A}\cdot\mathbf{B}| \leq |\mathbf{A}||\mathbf{B}|.$

Algebraically we get from the definition:
$|\mathbf{A}\cdot\mathbf{B}| = |\mathbf{A}||\mathbf{B}||\cos\theta| \leq |\mathbf{A}||\mathbf{B}|\cdot 1 = |\mathbf{A}||\mathbf{B}|$, where θ is the angle ($< 180°$) between vectors $\mathbf{A}$ and $\mathbf{B}$.

Geometrically we can see that the direct line between the origin and the end point of vector $\mathbf{A}+\mathbf{B}$ is obviously shorter than the path from origin to the end point of $\mathbf{A}$ and then from there to the end point of $\mathbf{A}+\mathbf{B}$. Thus $|\mathbf{A}+\mathbf{B}| \leq |\mathbf{A}| + |\mathbf{B}|$.

If $a, b \geq 0$ and $a^2 \leq b^2$ then $a \leq b$. Algebraically we get:

$$\begin{aligned}
|\mathbf{A}+\mathbf{B}|^2 = (\mathbf{A}+\mathbf{B}) \cdot (\mathbf{A}+\mathbf{B}) &= \mathbf{A} \cdot \mathbf{A} + 2\mathbf{A} \cdot \mathbf{B} + \mathbf{B} \cdot \mathbf{B} \\
&= |\mathbf{A}|^2 + 2\mathbf{A} \cdot \mathbf{B} + |\mathbf{B}|^2 \\
&\leq |\mathbf{A}|^2 + 2|\mathbf{A}||\mathbf{B}| + |\mathbf{B}|^2 \\
&= (|\mathbf{A}| + |\mathbf{B}|)^2
\end{aligned}$$

$\Rightarrow |\mathbf{A}+\mathbf{B}| \leq |\mathbf{A}| + |\mathbf{B}|$.