Analytical Mechanics

Exercises 1.9-1.16

(Exercise descriptions [with possible slight modifications] from Analytical Mechanics by Fowles and Cassiday, 7th International Student Edition. Solutions by: Waves and Tensors)

Exercise 1.9: Prove the vector identity
$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$.

Solution:

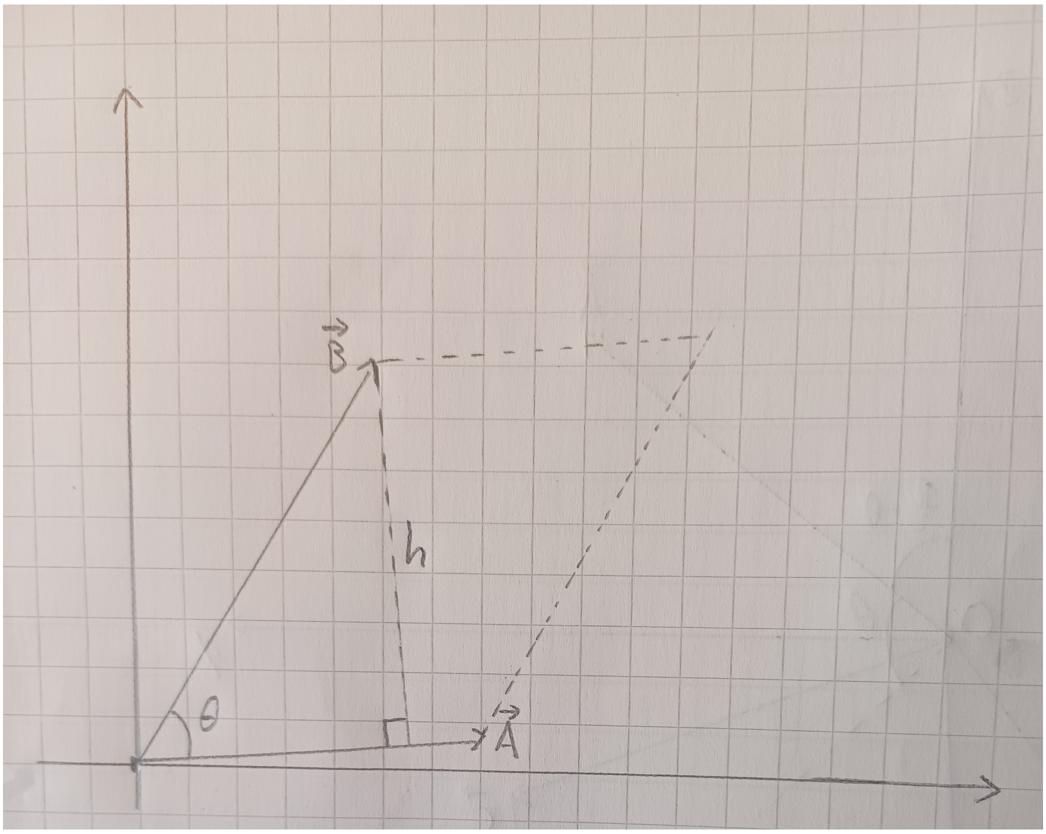
According to (1.5.1):
$\mathbf{B} \times \mathbf{C} = (B_y C_z - B_z C_y, B_z C_x - B_x C_z, B_x C_y - B_y C_x)$

Thus:

$$
\begin{aligned}
\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= (A_y(B_x C_y - B_y C_x) - A_z(B_z C_x - B_x C_z), \\
&\quad A_z(B_y C_z - B_z C_y) - A_x(B_x C_y - B_y C_x), \\
&\quad A_x(B_z C_x - B_x C_z) - A_y(B_y C_z - B_z C_y)) \\
&= (A_y B_x C_y - A_y B_y C_x - A_z B_z C_x - A_z B_x C_z, \\
&\quad A_z B_y C_z - A_z B_z C_y - A_x B_x C_y - A_x B_y C_x, \\
&\quad A_x B_z C_x - A_x B_x C_z - A_y B_y C_z - A_y B_z C_y) \\
&= ((A_x C_x + A_y C_y + A_z C_z) B_x - (A_x B_x + A_y B_y + A_z B_z) C_x, \\
&\quad (A_x C_x + A_y C_y + A_z C_z) B_y - (A_x B_x + A_y B_y + A_z B_z) C_y, \\
&\quad (A_x C_x + A_y C_y + A_z C_z) B_z - (A_x B_x + A_y B_y + A_z B_z) C_z) \\
&= (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C} \\
&= \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}).
\end{aligned}
$$

Exercise 1.10: Two vectors **A** and **B** represent concurrent sides of a parallelogram. Show that the area of the parallelogram is equal to $|\mathbf{A} \times \mathbf{B}|$.

Solution:



The area of the parallelogram is $A = |\mathbf{A}| \cdot h$. For the situation in the picture: $h = |\mathbf{B}| \sin\theta$, where $\theta = \angle(\mathbf{A}, \mathbf{B})$. Thus $A = |\mathbf{A}||\mathbf{B}| \sin\theta = |\mathbf{A} \times \mathbf{B}|$ (if $0 < \theta \leq 90°$). For $90° < \theta < 180°$ we put $\mathbf{A} \mapsto -\mathbf{A}$ to get the same result.

Exercise 1.11: Show that $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$ is not equal to $\mathbf{B} \cdot (\mathbf{A} \times \mathbf{C})$.

Solution:

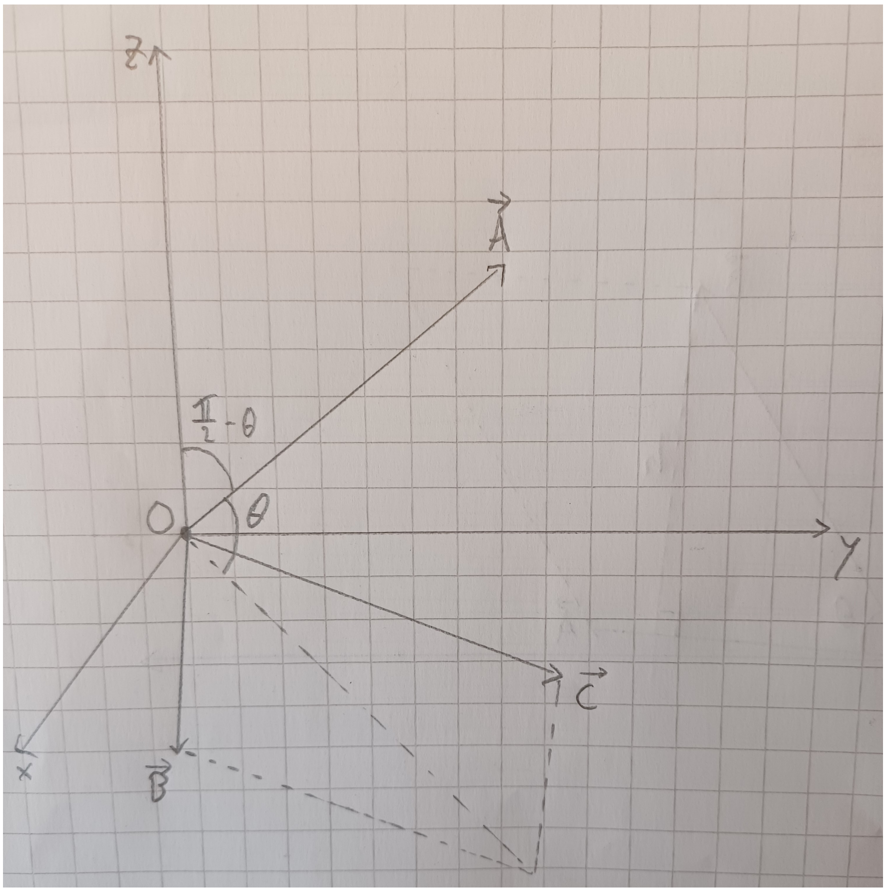
$$\begin{cases} \mathbf{A} = (1, 0, 0) \\ \mathbf{B} = (1, -1, 0) \\ \mathbf{C} = (0, 0, 1) \end{cases}$$

We get from (1.7.1) (and from Example (1.7.1)):

$$\begin{vmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -1 \neq +1 = \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

Exercise 1.12: Three vectors $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ represent three concurrent edges of a parallelepiped. Show that the volume of the parallelepiped is equal to $|\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})|$.

Solution:

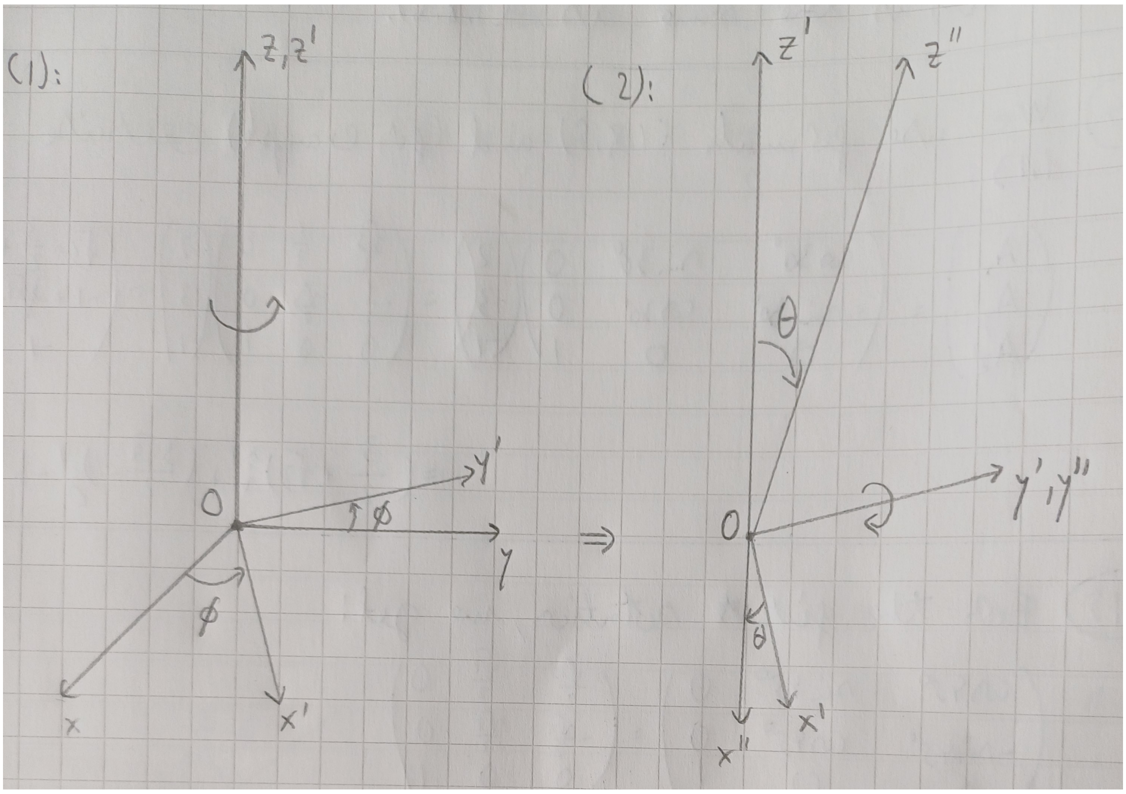


Let us assume that vectors $\mathbf{B}$ and $\mathbf{C}$ are in the xy-plane so that $\mathbf{B} \times \mathbf{C}$ is in the positive z-direction (as in the picture). Let θ be the angle between $\mathbf{A}$ and $\mathbf{B} + \mathbf{C}$ (which is in the xy-plane also).

The height of the parallelepiped is $h = |\mathbf{A}| \sin\theta$. The volume is thus $V = |\mathbf{B} \times \mathbf{C}| \cdot |\mathbf{A}| \sin\theta = |\mathbf{B} \times \mathbf{C}| \cdot |\mathbf{A}| \cos\left(\frac{\pi}{2} - \theta\right) = |\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})|$ for $0 < \theta \leq 90°$. For $90° < \theta < 180°$ we put $\mathbf{A} \mapsto -\mathbf{A}$ to get the same result. Also, if $\mathbf{B} \times \mathbf{C}$ is in the negative z-direction, we put $\mathbf{B} \times \mathbf{C} \mapsto -(\mathbf{B} \times \mathbf{C})$ to again get the same result, since the absolute value of the dot product does not change.

Exercise 1.13: Verify the transformation matrix for a rotation about the z-axis through an angle ϕ followed by a rotation about the y'-axis through an angle θ, as given in Example 1.8.2.

Solution:



From (1.8.5) we calculate the transformation matrix for situation (1):

$$\begin{pmatrix} \mathbf{i}\cdot\mathbf{i}' & \mathbf{j}\cdot\mathbf{i}' & \mathbf{k}\cdot\mathbf{i}' \\ \mathbf{i}\cdot\mathbf{j}' & \mathbf{j}\cdot\mathbf{j}' & \mathbf{k}\cdot\mathbf{j}' \\ \mathbf{i}\cdot\mathbf{k}' & \mathbf{j}\cdot\mathbf{k}' & \mathbf{k}\cdot\mathbf{k}' \end{pmatrix} = \begin{pmatrix} |\mathbf{i}|\cdot|\mathbf{i}'|\cos\phi & \cos\left(\frac{\pi}{2}-\phi\right) & \cos\frac{\pi}{2} \\ \cos\left(\frac{\pi}{2}+\phi\right) & \cos\phi & \cos\frac{\pi}{2} \\ \cos\frac{\pi}{2} & \cos\frac{\pi}{2} & \cos 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

For situation (2):

$$\begin{pmatrix} \mathbf{i}'\cdot\mathbf{i}'' & \mathbf{j}'\cdot\mathbf{i}'' & \mathbf{k}'\cdot\mathbf{i}'' \\ \mathbf{i}'\cdot\mathbf{j}'' & \mathbf{j}'\cdot\mathbf{j}'' & \mathbf{k}'\cdot\mathbf{j}'' \\ \mathbf{i}'\cdot\mathbf{k}'' & \mathbf{j}'\cdot\mathbf{k}'' & \mathbf{k}'\cdot\mathbf{k}'' \end{pmatrix} = \begin{pmatrix} |\mathbf{i}'|\cdot|\mathbf{i}''|\cos\theta & \cos\frac{\pi}{2} & \cos\left(\frac{\pi}{2}+\theta\right) \\ \cos\frac{\pi}{2} & \cos 0 & \cos\frac{\pi}{2} \\ \cos\left(\frac{\pi}{2}-\theta\right) & \cos\frac{\pi}{2} & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$

The full transformation matrix is thus:

$$\begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{pmatrix},$$

which is the same as in (1.8.6).

Exercise 1.14: Express the vector $2\mathbf{i}+3\mathbf{j}-\mathbf{k}$ in the primed triad $\mathbf{i}'\mathbf{j}'\mathbf{k}'$ in which the $x'y'$-axes are rotated about the z-axis (which coincides with the z'-axis) through an angle of 30°.

Solution:

We use formula (1.8.5) and (for example) exercise 1.13:

$$\begin{aligned}
\begin{pmatrix} A_{x'} \\ A_{y'} \\ A_{z'} \end{pmatrix} = \begin{pmatrix} \cos 30^\circ & \sin 30^\circ & 0 \\ -\sin 30^\circ & \cos 30^\circ & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} &= \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \\
&= \begin{pmatrix} \sqrt{3}+\frac{3}{2}+0 \\ -1+\frac{3\sqrt{3}}{2}+0 \\ -1 \end{pmatrix} \\
&= (\sqrt{3}+\frac{3}{2})\mathbf{i}' + (-1+\frac{3\sqrt{3}}{2})\mathbf{j}' - \mathbf{k}'.
\end{aligned}$$

Exercise 1.15: Consider two Cartesian coordinate systems xyz and $x'y'z'$ that initially coincide. The $x'y'z'$ undergoes three successive counterclockwise 45° rotations about the following axes: first, about the fixed z-axis; second, about its own x'-axis (which has now been rotated); finally, about its own z'-axis (which has also been rotated). Find the components of a unit vector $\mathbf{X}$ in the xyz coordinate system that points along the direction of the x'-axis in the rotated $x'y'z'$ system. (*Hint. It would be useful to find three transformation matrices that depict each of the above rotations. The resulting transformation matrix is simply their product.*)

Solution:

For the first rotation, we get:

$$\begin{pmatrix} \cos 45^\circ & \sin 45^\circ & 0 \\ -\sin 45^\circ & \cos 45^\circ & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

For the second rotation, we get:

$$\begin{pmatrix} \cos 0 & \cos\frac{\pi}{2} & \cos\frac{\pi}{2} \\ \cos\frac{\pi}{2} & \cos 45^\circ & \cos(90^\circ + 45^\circ) \\ \cos\frac{\pi}{2} & \cos(90^\circ - 45^\circ) & \cos 45^\circ \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$$

For the third rotation, we get:

$$\begin{pmatrix} \cos 45^\circ & \sin 45^\circ & 0 \\ -\sin 45^\circ & \cos 45^\circ & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The full transformation matrix R is thus:

$$R = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} - \frac{\sqrt{2}}{4} & \frac{1}{2} + \frac{\sqrt{2}}{4} & -\frac{1}{2} \\ -\frac{1}{2} - \frac{\sqrt{2}}{4} & -\frac{1}{2} + \frac{\sqrt{2}}{4} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$$

Now $\mathbf{X}' = R\mathbf{X}, \mathbf{X} = (x, y, z), \mathbf{X}' = (1, 0, 0)$ and we must have $|\mathbf{X}| = 1$.

$$\begin{cases} (1 - \frac{\sqrt{2}}{2})x + (1 + \frac{\sqrt{2}}{2})y - z = 2 \\ (-1 - \frac{\sqrt{2}}{2})x + (-1 + \frac{\sqrt{2}}{2})y - z = 0 \\ -x + y + \sqrt{2}z = 0 \end{cases}$$

Solving this set of linear equations we get:

$$\begin{cases} x = \frac{1}{2} - \frac{\sqrt{2}}{4} \\ y = \frac{1}{2} + \frac{\sqrt{2}}{4} \\ z = -\frac{1}{2} \end{cases}$$

The norm of $\mathbf{X}$ is now $|\mathbf{X}| = \sqrt{(\frac{1}{2} - \frac{\sqrt{2}}{4})^2 + (\frac{1}{2} + \frac{\sqrt{2}}{4})^2 + \frac{1}{4}} = 1$. The final answer is $\mathbf{X} = (\frac{1}{2} - \frac{\sqrt{2}}{4})\mathbf{i} + (\frac{1}{2} + \frac{\sqrt{2}}{4})\mathbf{j} - \frac{1}{2}\mathbf{k}$.

Exercise 1.16: A racing car moves on a circle of constant radius b. If the speed of the car varies with time t according to the equation $v = ct$, where c is a positive constant, show that the angle between the velocity vector and the acceleration vector is $45°$ at time $t = \sqrt{\frac{b}{c}}$. (*Hint. At this time the tangential and normal components of the acceleration are equal in magnitude.*)

Solution:

See figure 1.11.1 for polar coordinates.

$\mathbf{v}(t) = ct\mathbf{e}_\theta$,
$\mathbf{a}(t) = \frac{d\mathbf{v}(t)}{dt} = c\mathbf{e}_\theta + ct\frac{d\mathbf{e}_\theta}{dt} = c\mathbf{e}_\theta - ct\frac{d\theta}{dt}\mathbf{e}_r$.

Let ϕ be the angle between $\mathbf{v}$ and $\mathbf{a}$ at time $t = \sqrt{\frac{b}{c}}$. We must show that $\phi = 45° = \frac{\pi}{4}$.

$$\cos\phi = \frac{\mathbf{v}(\sqrt{\frac{b}{c}}) \cdot \mathbf{a}(\sqrt{\frac{b}{c}})}{|\mathbf{v}(\sqrt{\frac{b}{c}})||\mathbf{a}(\sqrt{\frac{b}{c}})|}$$

$$\mathbf{v}(\sqrt{\frac{b}{c}}) = c\sqrt{\frac{b}{c}}\mathbf{e}_\theta \Rightarrow |\mathbf{v}(\sqrt{\frac{b}{c}})| = \sqrt{bc},$$

$$\mathbf{a}(\sqrt{\frac{b}{c}}) = c\mathbf{e}_\theta - c\sqrt{\frac{b}{c}}\frac{d\theta}{dt}\mathbf{e}_r \Rightarrow |\mathbf{a}(\sqrt{\frac{b}{c}})| = \sqrt{c^2 + bc(\frac{d\theta}{dt})^2} = \sqrt{c^2 + c^2} = c\sqrt{2} \;\; (c > 0).$$

Thus we get:

$$\cos\phi = \frac{c^2 \cdot \sqrt{\frac{b}{c}}}{c\sqrt{2} \cdot \sqrt{bc}} = \frac{1}{\sqrt{2}} \cdot \sqrt{\frac{bc}{bc}} = \frac{1}{\sqrt{2}} \Rightarrow \phi = 45°.$$