Analytical Mechanics

Exercises 1.17-1.24

(Exercise descriptions [with possible slight modifications] from Analytical Mechanics by Fowles and Cassiday, 7th International Student Edition.
Solutions by: Waves and Tensors)

Exercise 1.17: A small ball is fastened to a long rubber band and twirled around in such a way that the ball moves in an elliptical path given by the equation

$\mathbf{r}(t) = \mathbf{i}b\cos(\omega t) + \mathbf{j}2b\sin(\omega t)$,

where b and ω are constants. Find the speed of the ball as a function of t. In particular, find v at $t = 0$ and $t = \frac{\pi}{2\omega}$, at which times the ball is, respectively, at its minimum and maximum distances from the origin.

Solution:

We get the velocity of the ball from equation (1.10.3):

$\mathbf{v} = \frac{d\mathbf{r}}{dt} = -\mathbf{i}b\omega\sin(\omega t) + \mathbf{j}2b\omega\cos(\omega t)$.

The speed is thus:

$$\begin{aligned} v = |\mathbf{v}| &= [(-b\omega\sin(\omega t))^2 + (2b\omega\cos(\omega t))^2]^{\frac{1}{2}} \\ &= [b^2\omega^2\sin^2(\omega t) + 4b^2\omega^2\cos^2(\omega t)]^{\frac{1}{2}} \\ &= |b| \cdot |\omega| \cdot [\sin^2(\omega t) + 4\cos^2(\omega t)]^{\frac{1}{2}}. \end{aligned}$$

At $t = 0$ we have $v = |b||\omega| \cdot [\sin^2 0 + 4\cos^2 0]^{\frac{1}{2}} = 2|b||\omega|$.

At $t = \frac{\pi}{2\omega}$ we have $v = |b||\omega| \cdot [\sin^2 \frac{\pi}{2} + 4\cos^2 \frac{\pi}{2}]^{\frac{1}{2}} = |b||\omega|$.

Exercise 1.18: A buzzing fly moves in a helical path given by the equation
$\mathbf{r}(t) = \mathbf{i}b\sin(\omega t) + \mathbf{j}b\cos(\omega t) + \mathbf{k}ct^2$.

Show that the magnitude of the acceleration of the fly is constant, provided b, ω and c are constant.

Solution:

The acceleration of the fly we get from equation (1.10.8):
$\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = -\mathbf{i}b\omega^2\sin(\omega t) - \mathbf{j}b\omega^2\cos(\omega t) + \mathbf{k}2c$.

The magnitude of the acceleration is:

$$\begin{aligned}
|\mathbf{a}| &= [(-b\omega^2\sin(\omega t))^2 + (-b\omega^2\cos(\omega t))^2 + (2c)^2]^{\frac{1}{2}} \\
&= [b^2\omega^4\sin^2(\omega t) + b^2\omega^4\cos^2(\omega t) + 4c^2]^{\frac{1}{2}} \\
&= [b^2\omega^4 + 4c^2]^{\frac{1}{2}} \\
&= \text{constant.}
\end{aligned}$$

Exercise 1.19: A bee goes out from its hive in a spiral path given in plane polar coordinates by

$r = be^{kt}, \quad \theta = ct,$

where b, k, and c are positive constants. Show that the angle between the velocity vector and the acceleration vector remains constant as the bee moves outward. (*Hint: Find* $\frac{\mathbf{v}\cdot\mathbf{a}}{va}$.)

Solution:

We get $r = be^{kt}, \dot{r} = bke^{kt}, \ddot{r} = bk^2e^{kt}, \theta = ct, \dot{\theta} = c$ and $\ddot{\theta} = 0$.
Plugging these values into equations (1.11.7) and (1.11.9) we get:

$\mathbf{v} = bke^{kt}\mathbf{e}_r + bce^{kt}\mathbf{e}_\theta$

$\mathbf{a} = (bk^2e^{kt} - be^{kt} \cdot c^2)\mathbf{e}_r + (be^{kt} \cdot 0 + 2bke^{kt} \cdot c)\mathbf{e}_\theta = (k^2 - c^2)be^{kt}\mathbf{e}_r + 2bcke^{kt}\mathbf{e}_\theta$

We get the cosine of the angle ϕ between $\mathbf{a}$ and $\mathbf{v}$ from equation (1.4.6):

$$
\begin{aligned}
\cos\phi = \frac{\mathbf{v}\cdot\mathbf{a}}{va} &= \frac{bke^{kt} \cdot (k^2 - c^2)be^{kt} + bce^{kt} \cdot 2bcke^{kt}}{[(bke^{kt})^2 + (bce^{kt})^2]^{\frac{1}{2}} \cdot [((k^2 - c^2)be^{kt})^2 + (2bcke^{kt})^2]^{\frac{1}{2}}} \\
&= \frac{(k^2 - c^2)b^2k + 2b^2c^2k}{[b^2k^2 + b^2c^2]^{\frac{1}{2}} \cdot [(k^2 - c^2)^2b^2 + 4b^2c^2k^2]^{\frac{1}{2}}} \\
&= \frac{(k^2 - c^2)k + 2c^2k}{[k^2 + c^2]^{\frac{1}{2}} \cdot [(k^2 - c^2)^2 + 4c^2k^2]^{\frac{1}{2}}} \\
&= \frac{(k^2 + c^2)k}{(k^2 + c^2)^{\frac{1}{2}} \cdot [(k^2 + c^2)^2]^{\frac{1}{2}}} \\
&= \frac{k}{\sqrt{k^2 + c^2}} \\
&= \text{constant.}
\end{aligned}
$$

Exercise 1.20: Work Exercise 1.18 using cylindrical coordinates where $R = b, \phi = \omega t$, and $z = ct^2$.

Solution:

We get $R = b, \dot{R} = 0, \ddot{R} = 0, \phi = \omega t, \dot{\phi} = \omega, \ddot{\phi} = 0, z = ct^2, \dot{z} = 2ct$ and $\ddot{z} = 2c$. We get the acceleration of the fly from equation (1.12.3):

$\mathbf{a} = (0 - b \cdot \omega^2)\mathbf{e}_R + (2 \cdot 0 \cdot \omega + b \cdot 0)\mathbf{e}_\phi + 2c\mathbf{e}_z = -b\omega^2\mathbf{e}_R + 2c\mathbf{e}_z$

The magnitude of the acceleration of the fly is thus:

$|\mathbf{a}| = [(-b\omega^2)^2 + (2c)^2]^{\frac{1}{2}} = [b^2\omega^4 + 4c^2]^{\frac{1}{2}}$,

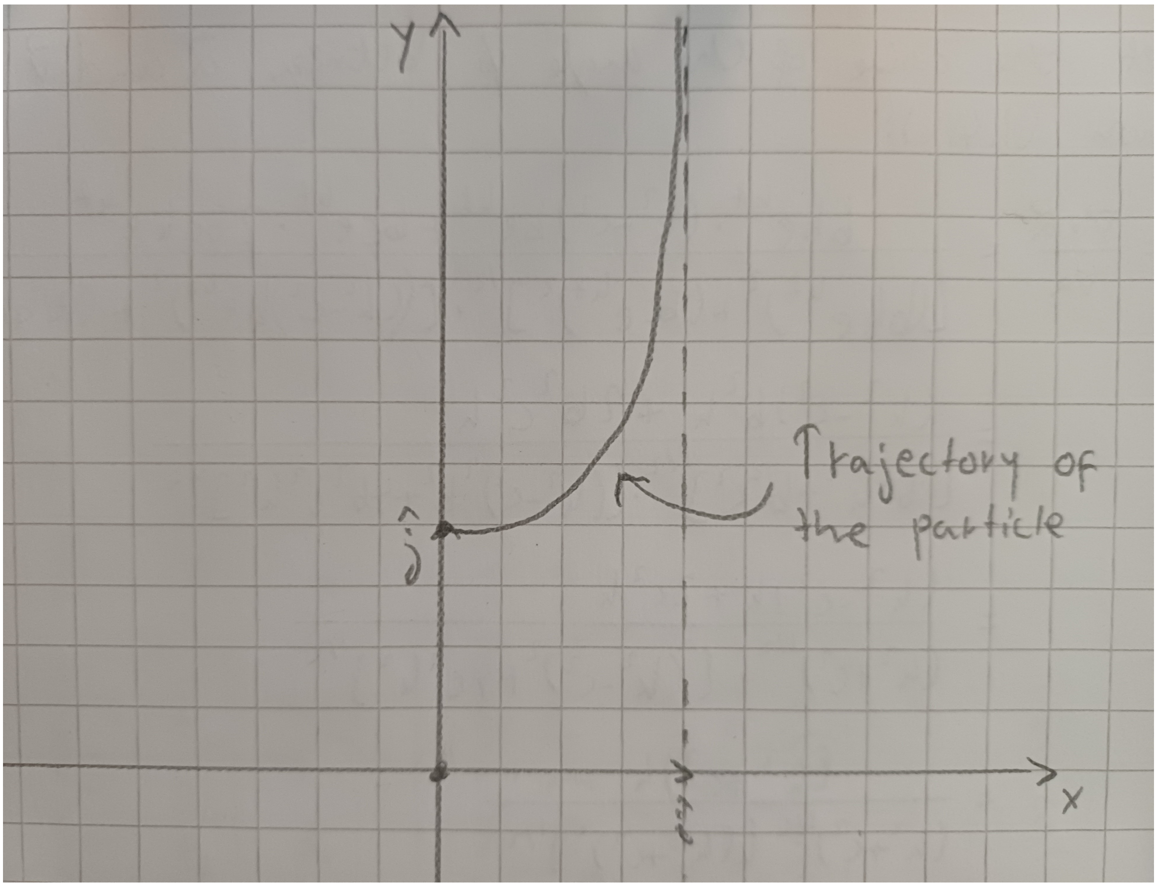
which is the same result as in Exercise 1.18.

Exercise 1.21: The position of a particle as a function of time is given by

$\mathbf{r}(t) = \mathbf{i}(1 - e^{-kt}) + \mathbf{j}e^{kt}$,

where k is a positive constant. Find the velocity and acceleration of the particle. Sketch its trajectory.

Solution:



$\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt} = ke^{-kt}\mathbf{i} + ke^{kt}\mathbf{j}$

$\mathbf{a}(t) = \frac{d\mathbf{v}(t)}{dt} = -k^2e^{-kt}\mathbf{i} + k^2e^{kt}\mathbf{j}$

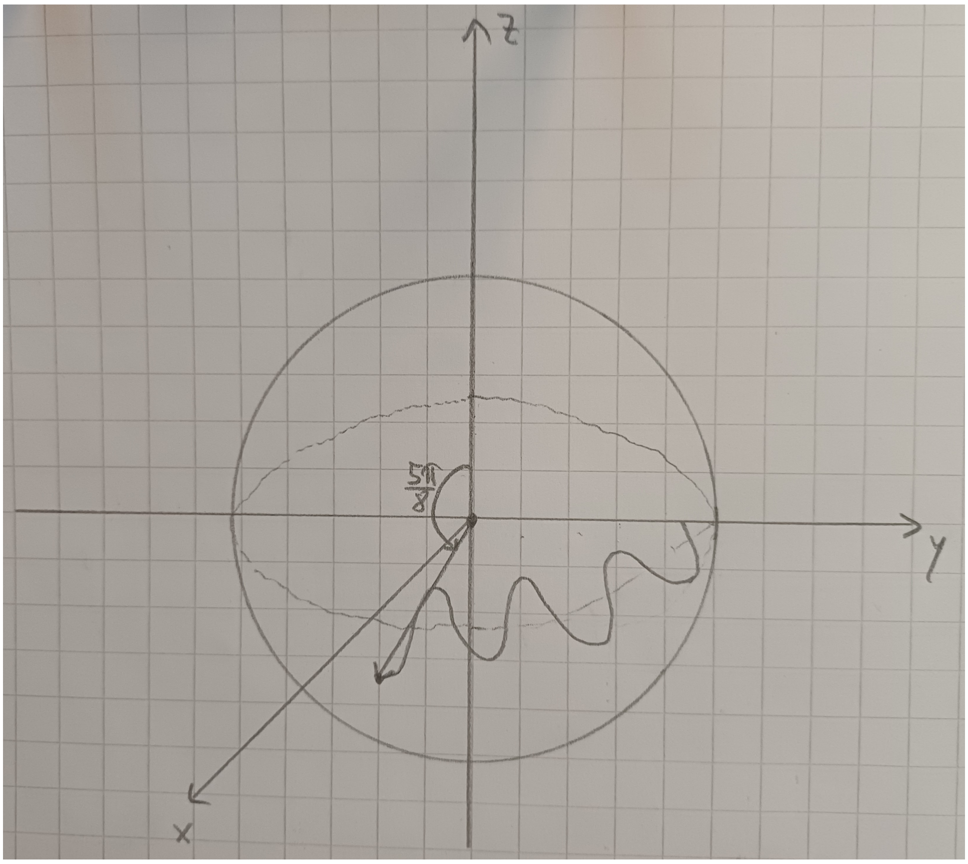
$\mathbf{r}(0) = \mathbf{j}$ and $\mathbf{r}(t \to \infty) = \mathbf{i} + \infty \cdot \mathbf{j}$.

Exercise 1.22: An ant crawls on the surface of a ball of radius b in such a manner that the ant's motion is given in spherical coordinates by the equations

$r = b, \quad \phi = \omega t, \quad \theta = \frac{\pi}{2}[1 + \frac{1}{4}\cos(4\omega t)].$

Find the speed of the ant as a function of the time t. What sort of path is represented by the above equations?

Solution:



We get $r = b, \dot{r} = 0, \phi = \omega t, \dot{\phi} = \omega, \theta = \frac{\pi}{2}[1 + \frac{1}{4}\cos(4\omega t)], \dot{\theta} = -\frac{\pi}{2}\omega \sin(4\omega t)$.

We get the velocity of the ant from equation (1.12.12):

$\mathbf{v}(t) = b\omega \sin(\frac{\pi}{2}[1 + \frac{1}{4}\cos(4\omega t)])\mathbf{e}_\phi - \frac{\pi}{2} b\omega \sin(4\omega t)\mathbf{e}_\theta$.

So the speed of the ant is:

$$\begin{aligned}
|\mathbf{v}(t)| &= [b^2\omega^2 \sin^2(\frac{\pi}{2}[1+\frac{1}{4}\cos(4\omega t)]) + \frac{\pi^2}{4}b^2\omega^2\sin^2(4\omega t)]^{\frac{1}{2}} \\
&= b|\omega|[\sin^2(\frac{\pi}{2}[1+\frac{1}{4}\cos(4\omega t)]) + \frac{\pi^2}{4}\sin^2(4\omega t)]^{\frac{1}{2}}.
\end{aligned}$$

At $t = 0$: $r = b, \phi = 0$ and $\theta = \frac{\pi}{2}(1 + \frac{1}{4}) = \frac{5\pi}{8}$.

r does not change so the ant is always on the surface of the ball. ϕ increases (assuming $\omega > 0$) linearly with time t whereas θ changes between a minimum of $\frac{\pi}{2}(1 - \frac{1}{4}) = \frac{3\pi}{8}$ and a maximum of $\frac{5\pi}{8}$. θ changes periodically in a cosinesoidal way.

Exercise 1.23: Prove that $\mathbf{v} \cdot \mathbf{a} = v\dot{v}$ and, hence, that for a moving particle $\mathbf{v}$ and $\mathbf{a}$ are perpendicular to each other if the speed v is constant. (*Hint: Differentiate both sides of the equation* $\mathbf{v} \cdot \mathbf{v} = v^2$ *with respect to* t. *Note,* $\dot{v}$ *is not the same as* $|\mathbf{a}|$. *It is the magnitude of the acceleration of the particle along its instantaneous direction of motion.*)

Solution:

From equation (1.9.5) we get:
$v^2 = \mathbf{v} \cdot \mathbf{v} \Rightarrow 2v\dot{v} = \frac{d\mathbf{v}}{dt} \cdot \mathbf{v} + \mathbf{v} \cdot \frac{d\mathbf{v}}{dt} \Rightarrow \mathbf{v} \cdot \mathbf{a} = v\dot{v}$.

If $v =$ constant then $\dot{v} = 0$ and $\mathbf{v} \cdot \mathbf{a} = 0$ so $\mathbf{v}$ and $\mathbf{a}$ are perpendicular.

Exercise 1.24: Prove that
$\frac{d}{dt}[\mathbf{r} \cdot (\mathbf{v} \times \mathbf{a})] = \mathbf{r} \cdot (\mathbf{v} \times \dot{\mathbf{a}})$.

Solution:

We get from equations (1.9.5) and (1.9.6):

$$\begin{aligned}
\frac{d}{dt}[\mathbf{r} \cdot (\mathbf{v} \times \mathbf{a})] &= \frac{d\mathbf{r}}{dt} \cdot (\mathbf{v} \times \mathbf{a}) + \mathbf{r} \cdot \frac{d}{dt}(\mathbf{v} \times \mathbf{a}) \\
&= \mathbf{v} \cdot (\mathbf{v} \times \mathbf{a}) + \mathbf{r} \cdot (\frac{d\mathbf{v}}{dt} \times \mathbf{a} + \mathbf{v} \times \frac{d\mathbf{a}}{dt}) \\
&= 0 + \mathbf{r} \cdot (0 + \mathbf{v} \times \dot{\mathbf{a}}) \\
&= \mathbf{r} \cdot (\mathbf{v} \times \dot{\mathbf{a}}).
\end{aligned}$$