Analytical Mechanics

Exercises 1.25-1.30

(Exercise descriptions [with possible slight modifications] from Analytical Mechanics by Fowles and Cassiday, 7th International Student Edition. Solutions by: Waves and Tensors)

Exercise 1.25: Show that the tangential component of the acceleration of a moving particle is given by the expression
$a_\tau = \frac{\mathbf{v}\cdot\mathbf{a}}{v}$,
and the normal component is therefore
$a_n = (a^2 - a_\tau^2)^{\frac{1}{2}} = [a^2 - \frac{(\mathbf{v}\cdot\mathbf{a})^2}{v^2}]^{\frac{1}{2}}$.

Solution:

The acceleration $\mathbf{a} = \mathbf{a}_n + \mathbf{a}_\tau$ and according to the book (page 32), $\mathbf{v}\cdot\mathbf{a}_n = 0$ and $\mathbf{v}\cdot\mathbf{a}_\tau = va_\tau\cos 0° = va_\tau$. Thus:

$\mathbf{v}\cdot\mathbf{a} = \mathbf{v}\cdot\mathbf{a}_n + \mathbf{v}\cdot\mathbf{a}_\tau = 0 + va_\tau \Rightarrow a_\tau = \frac{\mathbf{v}\cdot\mathbf{a}}{v}$.

Because $\mathbf{a}_n \perp \mathbf{a}_\tau$ we get:

$a^2 = \mathbf{a}\cdot\mathbf{a} = (\mathbf{a}_n + \mathbf{a}_\tau)\cdot(\mathbf{a}_n + \mathbf{a}_\tau) = a_n^2 + 2\mathbf{a}_n\cdot\mathbf{a}_\tau + a_\tau^2 = a_n^2 + a_\tau^2$
$\Rightarrow a_n = \pm(a^2 - a_\tau^2)^{\frac{1}{2}} = [a^2 - \frac{(\mathbf{v}\cdot\mathbf{a})^2}{v^2}]^{\frac{1}{2}}$.

Exercise 1.26: Use the above result to find the tangential and normal components of the acceleration as functions of time in Exercises 1.18 and 1.19.

Solution:

For Exercise 1.18:

$\mathbf{r}(t) = \mathbf{i}b\sin\omega t + \mathbf{j}b\cos\omega t + \mathbf{k}ct^2,$
$\mathbf{v}(t) = \mathbf{i}\omega b\cos\omega t - \mathbf{j}\omega b\sin\omega t + \mathbf{k}2ct,$
$\mathbf{a}(t) = -\mathbf{i}\omega^2 b\sin\omega t - \mathbf{j}\omega^2 b\cos\omega t + \mathbf{k}2c.$

$$\begin{aligned} a_\tau = \frac{\mathbf{v}\cdot\mathbf{a}}{v} &= \frac{-b^2\omega^3\cos\omega t\sin\omega t + b^2\omega^3\cos\omega t\sin\omega t + 4c^2t}{(b^2\omega^2\cos^2\omega t + b^2\omega^2\sin^2\omega t + 4c^2t^2)^{\frac{1}{2}}} \\ &= \frac{4c^2t}{[b^2\omega^2 + 4c^2t^2]^{\frac{1}{2}}}, \end{aligned}$$

$$\begin{aligned} a_n = (a^2 - a_\tau^2)^{\frac{1}{2}} &= [b^2\omega^4\sin^2\omega t + b^2\omega^4\cos^2\omega t + 4c^2 - \frac{16c^4t^2}{b^2\omega^2 + 4c^2t^2}]^{\frac{1}{2}} \\ &= [\frac{(b^2\omega^4 + 4c^2)(b^2\omega^2 + 4c^2t^2) - 16c^4t^2}{b^2\omega^2 + 4c^2t^2}]^{\frac{1}{2}} \\ &= [\frac{b^4\omega^6 + 4c^2b^2\omega^4t^2 + 4c^2b^2\omega^2 + 16c^4t^2 - 16c^4t^2}{b^2\omega^2 + 4c^2t^2}]^{\frac{1}{2}} \\ &= b\omega[\frac{b^2\omega^4 + 4c^2\omega^2t^2 + 4c^2}{b^2\omega^2 + 4c^2t^2}]^{\frac{1}{2}} \\ &= b\omega[\omega^2 + \frac{4c^2}{b^2\omega^2 + 4c^2t^2}]^{\frac{1}{2}}. \end{aligned}$$

For Exercise 1.19:

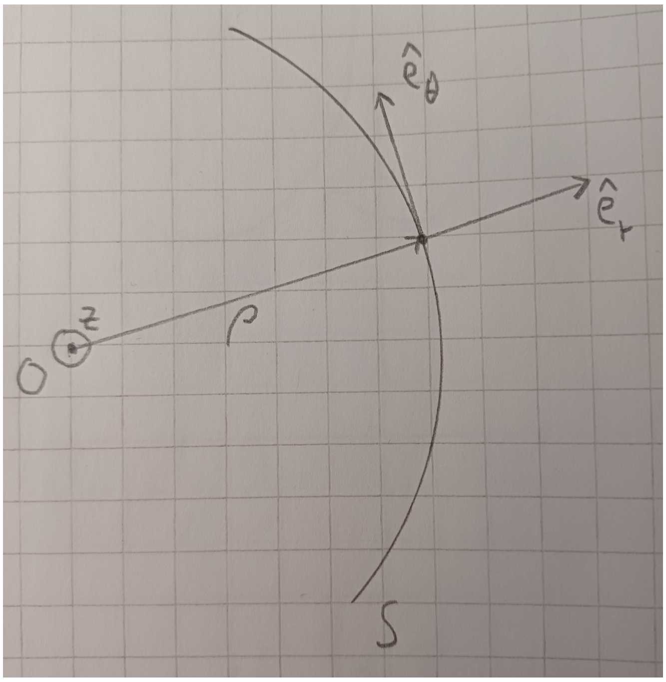
$\mathbf{v}(t) = bke^{kt}\mathbf{e}_r + bce^{kt}\mathbf{e}_\theta,$
$\mathbf{a}(t) = (k^2 - c^2)be^{kt}\mathbf{e}_r + 2bcke^{kt}\mathbf{e}_\theta.$

$$\begin{aligned} a_\tau = \frac{bke^{kt}\cdot(k^2 - c^2)be^{kt} + 2b^2c^2ke^{2kt}}{(b^2k^2e^{2kt} + b^2c^2e^{2kt})^{\frac{1}{2}}} &= \frac{b^2(k^2 - c^2)ke^{2kt} + 2b^2c^2ke^{2kt}}{(k^2 + c^2)^{\frac{1}{2}}be^{kt}} \\ &= bke^{kt}\cdot\frac{k^2 - c^2 + 2c^2}{(k^2 + c^2)^{\frac{1}{2}}} \\ &= bke^{kt}[k^2 + c^2]^{\frac{1}{2}}, \end{aligned}$$

$$
\begin{aligned}
a_n &= [(k^2 - c^2)^2 b^2 e^{2kt} + 4b^2 c^2 k^2 e^{2kt} - b^2 k^2 e^{2kt}(k^2 + c^2)]^{\frac{1}{2}} \\
&= be^{kt}[(k^2 - c^2)^2 + 4c^2 k^2 - (k^2 + c^2)k^2]^{\frac{1}{2}} \\
&= be^{kt}[k^4 - 2k^2 c^2 + c^4 + 4c^2 k^2 - k^4 - k^2 c^2]^{\frac{1}{2}} \\
&= bce^{kt}[k^2 + c^2]^{\frac{1}{2}}.
\end{aligned}
$$

Exercise 1.27: Prove that $|\mathbf{v} \times \mathbf{a}| = v^3/\rho$, where ρ is the radius of curvature of the path of a moving particle.

Solution:



For the curve S we have $r = \rho = \text{constant} \Rightarrow \dot{r} = \ddot{r} = 0$. From equations (1.11.7) and (1.11.9) we have (positive z-axis out of paper towards the viewer):
$\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta$
$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{e}_\theta$

Thus:

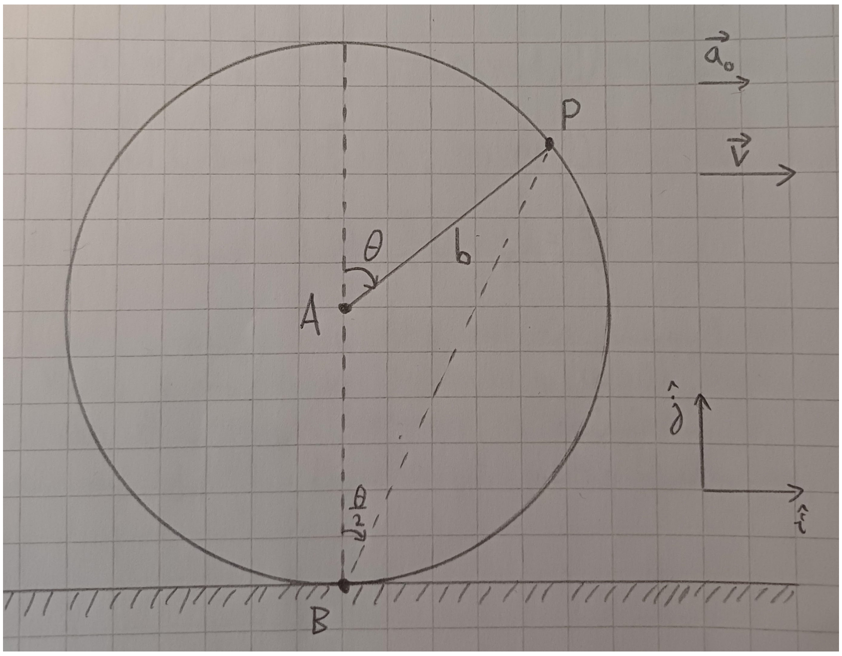
$$\begin{aligned} \mathbf{v} \times \mathbf{a} &= (r\dot{\theta}\mathbf{e}_\theta \times (-r\dot{\theta}^2)\mathbf{e}_r) + (r\dot{\theta}\mathbf{e}_\theta \times r\ddot{\theta}\mathbf{e}_\theta) \\ &= r^2\dot{\theta}^3(\mathbf{e}_r \times \mathbf{e}_\theta) + 0 \\ &= r^2\dot{\theta}^3\mathbf{e}_z \end{aligned}$$

$\Rightarrow |\mathbf{v} \times \mathbf{a}| = \rho^2\dot{\theta}^3 \quad (*)$

Now we see that $\mathbf{v} = \rho\dot{\theta}\mathbf{e}_\theta \Rightarrow v = \rho\dot{\theta} \Rightarrow \dot{\theta} = \frac{v}{\rho}$. Plugging into (*) we get $|\mathbf{v} \times \mathbf{a}| = \frac{v^3}{\rho}$.

Exercise 1.28: A wheel of radius b rolls along the ground with constant forward acceleration a_0. Show that, at any given instant, the magnitude of the acceleration of any point on the wheel is $(a_0^2 + v^4/b^2)^{\frac{1}{2}}$ relative to the center of the wheel and is also $a_0[2 + 2\cos\theta + v^4/a_0^2b^2 - (2v^2/a_0b)\sin\theta]^{\frac{1}{2}}$ relative to the ground. Here v is the instantaneous forward speed, and θ defines the location of the point on the wheel, measured forward from the highest point. Which point has the greatest acceleration relative to the ground?

Solution:



$\mathbf{r}_1 = (b\theta, b)$
$\mathbf{r}_2 = (b\sin\theta, b\cos\theta)$

With respect to point A, we calculate $|\ddot{\mathbf{r}}_2|$:

$\dot{\mathbf{r}}_2 = (b\dot{\theta}\cos\theta, -b\dot{\theta}\sin\theta)$
$\ddot{\mathbf{r}}_2 = (b\ddot{\theta}\cos\theta - b\dot{\theta}^2\sin\theta, -b\ddot{\theta}\sin\theta - b\dot{\theta}^2\cos\theta)$

We know that for $\theta = 0$ we have $\mathbf{v} = \dot{\mathbf{r}}_2 = (b\dot{\theta}, 0)$ so $v = b\dot{\theta}$ and thus $a_0 = \dot{v} = b\ddot{\theta}$. We get:

$\ddot{\mathbf{r}}_2 = (a_0\cos\theta - \frac{v^2}{b}\sin\theta, -a_0\sin\theta - \frac{v^2}{b}\cos\theta)$

Finally:

$$
\begin{aligned}
|\ddot{\mathbf{r}}_2| &= [(a_0 \cos\theta - \frac{v^2}{b}\sin\theta)^2 + (-a_0\sin\theta - \frac{v^2}{b}\cos\theta)^2]^{\frac{1}{2}} \\
&= [a_0^2\cos^2\theta - \frac{2v^2 a_0}{b}\cos\theta\sin\theta + \frac{v^4}{b^2}\sin^2\theta + a_0^2\sin^2\theta + \frac{2v^2a_0}{b}\cos\theta\sin\theta + \frac{v^4}{b^2}\cos^2\theta]^{\frac{1}{2}} \\
&= [a_0^2 + \frac{v^4}{b^2}]^{\frac{1}{2}}.
\end{aligned}
$$

With respect to point B, we calculate:

$$
\mathbf{r} = \mathbf{r}_1 + \mathbf{r}_2 \Rightarrow \ddot{\mathbf{r}} = \ddot{\mathbf{r}}_1 + \ddot{\mathbf{r}}_2 = (b\ddot{\theta}, 0) + \ddot{\mathbf{r}}_2 = (a_0 + a_0\cos\theta - \frac{v^2}{b}\sin\theta, -a_0\sin\theta - \frac{v^2}{b}\cos\theta)
$$

Thus we get the acceleration at point P relative to point B as:

$$
\begin{aligned}
|\ddot{\mathbf{r}}| &= [(a_0 + a_0\cos\theta - \frac{v^2}{b}\sin\theta)^2 + (-a_0\sin\theta - \frac{v^2}{b}\cos\theta)^2]^{\frac{1}{2}} \\
&= [a_0^2(1+\cos\theta)^2 - \frac{2v^2a_0}{b}(1+\cos\theta)\sin\theta + \frac{v^2}{b}\sin^2\theta + a_0^2\sin^2\theta + \frac{2v^2a_0}{b}\cos\theta\sin\theta \\
&+ \frac{v^2}{b}\cos^2\theta]^{\frac{1}{2}} \\
&= [a_0^2 + 2a_0^2\cos\theta + a_0^2\cos^2\theta + \frac{v^4}{b^2} - \frac{2v^2a_0}{b}\sin\theta + a_0^2\sin^2\theta]^{\frac{1}{2}} \\
&= a_0[2 + 2\cos\theta + \frac{v^4}{a_0^2b^2} - \frac{2v^2}{a_0 b}\sin\theta]^{\frac{1}{2}}.
\end{aligned}
$$

To find the maximum of this last acceleration, we have to find the maximum of function $f(\theta) = \cos\theta - \frac{v^2}{a_0 b}\sin\theta$ for $0 \le \theta < \infty$.

$$
\begin{aligned}
f'(\theta) &= -\sin\theta - \frac{v^2}{a_0 b}\cos\theta \\
f''(\theta) &= -\cos\theta + \frac{v^2}{a_0 b}\sin\theta = -f(\theta)
\end{aligned}
$$

The derivative is zero when:

$f'(\theta) = 0 \Rightarrow \sin\theta_n = -\frac{v^2}{a_0 b}\cos\theta_n \Rightarrow \theta_n = \arctan(-\frac{v^2}{a_0 b}) + n\pi, \quad n \in \mathbb{N}$.

Arctan goes from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$. Thus $-\frac{\pi}{2} < \theta_0 < 0 \Rightarrow \frac{\pi}{2} < \theta_1 < \pi$. For θ_1: $\cos\theta_1 < 0$ and $\sin\theta_1 > 0$ and so $f(\theta_1) < 0$ and $f''(\theta_1) > 0$. Thus θ_1 is a minimum. Due to the periodicity (period is $\le 2\pi$) of f, acceleration reaches a maximum when $\theta_{2n} = \arctan(-\frac{v^2}{a_0 b}) + 2n\pi, \quad n \in \mathbb{N} \setminus \{0\}$.

Exercise 1.29: What is the value of x ($\in \mathbb{R}$) that makes of following transformation $\mathbf{R}$ orthogonal?

$$\mathbf{R} = \begin{pmatrix} x & x & 0 \\ -x & x & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

What transformation is represented by an orthogonal $\mathbf{R}$?

Solution:

$$\mathbf{R}\tilde{\mathbf{R}} = \begin{pmatrix} x & x & 0 \\ -x & x & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x & -x & 0 \\ x & x & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2x^2 & 0 & 0 \\ 0 & 2x^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \tilde{\mathbf{R}}\mathbf{R}$$

For $\mathbf{R}$ to be orthogonal, $\mathbf{R}\tilde{\mathbf{R}} = I = \tilde{\mathbf{R}}\mathbf{R}$, so:

$$\begin{pmatrix} 2x^2 & 0 & 0 \\ 0 & 2x^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Leftrightarrow 2x^2 = 1 \Leftrightarrow x = \pm\frac{1}{\sqrt{2}}.$$

For $x = \frac{1}{\sqrt{2}}$:

$$\mathbf{R} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos 45^\circ & \sin 45^\circ & 0 \\ -\sin 45^\circ & \cos 45^\circ & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which is a 45° counter-clockwise rotation about the z-axis.

For $x = -\frac{1}{\sqrt{2}}$:

$$\mathbf{R} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos(-135^\circ) & \sin(-135^\circ) & 0 \\ -\sin(-135^\circ) & \cos(-135^\circ) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which is a 135° clockwise rotation about the z-axis.

Exercise 1.30: Use vector algebra to derive the following trigonometric identities
(a) $\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$,
(b) $\sin(\theta - \phi) = \sin\theta\cos\phi - \cos\theta\sin\phi$.

Solution:

We know that a rotation with respect to z-axis counter-clockwise by an angle $\theta - \phi$ is given by the transformation matrix $R_z(\theta - \phi) = R_z(\theta)R_z(-\phi)$:

$$\begin{pmatrix} \cos(\theta-\phi) & \sin(\theta-\phi) & 0 \\ -\sin(\theta-\phi) & \cos(\theta-\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \cos\theta\cos\phi + \sin\theta\sin\phi & \sin\theta\cos\phi - \cos\theta\sin\phi & 0 \\ -\sin\theta\cos\phi + \cos\theta\sin\phi & \cos\theta\cos\phi + \sin\theta\sin\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (*)$$

(a) (*) $\Rightarrow \cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$.

(b) (*) $\Rightarrow \sin(\theta - \phi) = \sin\theta\cos\phi - \cos\theta\sin\phi$.