Analytical Mechanics

Exercises 2.1-2.7

(Exercise descriptions [with possible slight modifications] from Analytical Mechanics by Fowles and Cassiday, 7th International Student Edition. Solutions by: Waves and Tensors)

Exercise 2.1: Find the velocity $\dot{x}$ and the position x as functions of the time t for a particle of mass m, which starts from rest at $x = 0$ and $t = 0$, subject to the following force functions:
(a) $F_x = F_0 + ct$,
(b) $F_x = F_0 \sin ct$,
(c) $F_x = F_0 e^{ct}$,
where F_0 and c are positive constants.

Solution:

We have $x_0 = 0, t_0 = 0$ and $v_0 = 0$. We use formula (2.2.1) and integrate.

(a)

$$m\ddot{x} = F_x = F_0 + ct \;\Rightarrow\; m\frac{d\dot{x}}{dt} = F_0 + ct = m\frac{dv}{dt}$$
$$\Rightarrow \int_{v_0}^{v} dv' = \frac{1}{m}\int_{t_0}^{t} (F_0 + ct')dt'$$
$$\Rightarrow v - v_0 = \frac{1}{m}(F_0 t + \frac{1}{2}ct^2 - F_0 t_0 - \frac{1}{2}ct_0^2)$$
$$\Rightarrow \dot{x}(t) = \frac{1}{m}(F_0 t + \frac{1}{2}ct^2), \text{ for } t \geq 0.$$

We integrate again to get the position:

$$\frac{dx}{dt} = \frac{1}{m}(F_0 t + \frac{1}{2}ct^2) \;\Rightarrow\; \int_{x_0}^{x} dx' = \frac{1}{m}\int_{t_0}^{t} (F_0 t' + \frac{1}{2}c(t')^2)dt'$$
$$\Rightarrow x(t) - x_0 = \frac{1}{m}(\frac{1}{2}F_0 t^2 + \frac{1}{6}ct^3 - \frac{1}{2}F_0 t_0^2 - \frac{1}{6}ct_0^3)$$
$$\Rightarrow x(t) = \frac{1}{2m}(F_0 t^2 + \frac{1}{3}ct^3), \text{ for } t \geq 0.$$

(b)

$$m\frac{dv}{dt} = F_0 \sin(ct) \;\Rightarrow\; \int_{v_0}^{v} dv' = \frac{F_0}{m}\int_{t_0}^{t} \sin(ct')dt'$$
$$\Rightarrow v - v_0 = \frac{F_0}{m}\cdot(-\frac{1}{c})(\cos(ct) - \cos(ct_0))$$
$$\Rightarrow \dot{x}(t) = \frac{F_0}{mc}(1 - \cos(ct)), \text{ for } t \geq 0.$$

We integrate again to get the position:

$$\frac{dx}{dt} = \frac{F_0}{mc}(1 - \cos{(ct)}) \;\;\Rightarrow\;\; x(t) - x_0 = \frac{F_0}{mc}(t - t_0 - \frac{1}{c}\sin{(ct)} + \frac{1}{c}\sin{(ct_0)})$$
$$\Rightarrow\;\; x(t) = \frac{F_0}{mc}(t - \frac{1}{c}\sin{(ct)}),\; \text{for } t \geq 0.$$

(c)

$$m\frac{dv}{dt} = F_0 e^{ct} \;\;\Rightarrow\;\; v - v_0 = \frac{F_0}{m}\int_{t_0}^{t} e^{ct'}dt' = \frac{F_0}{mc}(e^{ct} - e^{ct_0})$$
$$\Rightarrow\;\; \dot{x}(t) = \frac{F_0}{mc}(e^{ct} - 1),\; \text{for } t \geq 0.$$

We integrate again to get the position:

$$\frac{dx}{dt} = \frac{F_0}{mc}(e^{ct} - 1) \;\;\Rightarrow\;\; x(t) - x_0 = \frac{F_0}{mc}(\frac{1}{c}(e^{ct} - e^{ct_0}) - t + t_0)$$
$$\Rightarrow\;\; x(t) = \frac{F_0}{mc^2}(e^{ct} - 1 - ct),\; \text{for } t \geq 0.$$

Exercise 2.2: Find the velocity $\dot{x}$ as a function of the displacement x for a particle of mass m, which starts from rest at $x = 0$, subject to the following force functions:
(a) $F_x = F_0 + cx$,
(b) $F_x = F_0 e^{-cx}$,
(c) $F_x = F_0 \cos cx$,
where F_0 and c are positive constants.

Solution:

We have $x_0 = 0$ and $v_0 = 0$. We use formulas (2.3.1) and (2.3.2) and integrate. For each case we also have to find out when the solution is valid.
$F_x = F(x) = m\ddot{x} = m\frac{d\dot{x}}{dt} = m\frac{dx}{dt}\frac{d\dot{x}}{dx} = mv\frac{dv}{dx}$.
(a)

$$\begin{aligned}
F_x = F_0 + cx \quad &\Rightarrow \quad mv\frac{dv}{dx} = F_0 + cx \\
&\Rightarrow \quad \int_{v_0}^{v} v'dv' = \frac{1}{m}\int_{x_0}^{x}(F_0 + cx')dx' \\
&\Rightarrow \quad \frac{1}{2}v^2 - \frac{1}{2}v_0^2 = \frac{1}{m}(F_0x + \frac{1}{2}cx^2 - F_0x_0 - \frac{1}{2}cx_0^2) \\
&\Rightarrow \quad \dot{x}(x) = \pm(\frac{1}{m})^{\frac{1}{2}}(2F_0x + cx^2)^{\frac{1}{2}}.
\end{aligned}$$

The solution is valid when $f(x) = cx^2 + 2F_0x \geq 0$. $f(x)$ is an upwards opening parabola, which has zero points at $x = 0$ and $x = -\frac{2F_0}{c}$. Thus the solution is valid when $x \leq -\frac{2F_0}{c}$ or $x \geq 0$.

(b)

$$\begin{aligned}
F_x = F_0e^{-cx} \quad &\Rightarrow \quad \frac{1}{2}mv^2 = -\frac{F_0}{c}e^{-cx} + \frac{F_0}{c}e^{-cx_0} \\
&\Rightarrow \quad \dot{x}(x) = \pm(\frac{2F_0}{mc})^{\frac{1}{2}}(1 - e^{-cx})^{\frac{1}{2}}.
\end{aligned}$$

The solution is valid when $f(x) = 1 - e^{-cx} \geq 0$. Since the natural logarithm $\ln x$ is an increasing function, this implies that
$1 - e^{-cx} \geq 0 \Rightarrow e^{-cx} \leq 1 \Rightarrow \ln(e^{-cx}) \leq 0 \Rightarrow -cx \leq 0 \Rightarrow x \geq 0$. Thus the solution is valid when $x \geq 0$.

(c)

$$
\begin{aligned}
F_x = F_0 \cos(cx) \quad &\Rightarrow \quad \frac{1}{2}mv^2 = \int_{x_0}^{x} F_0 \cos(cx')dx' \\
&\Rightarrow \quad \frac{1}{2}mv^2 = F_0(\frac{1}{c}\sin(cx) - \frac{1}{c}\sin(cx_0)) \\
&\Rightarrow \quad \dot{x}(x) = \pm(\frac{2F_0}{mc})^{\frac{1}{2}}(\sin(cx))^{\frac{1}{2}}.
\end{aligned}
$$

The solution is valid when $f(x) = \sin(cx) \geq 0$. This implies that $0 + 2\pi n \leq cx \leq \pi + 2\pi n$ for $n \in \{0, 1, 2, \ldots\}$. Thus the solution is valid when $\frac{2\pi n}{c} \leq x \leq \frac{(1+2n)\pi}{c}$ and $n \in \{0, 1, 2, \ldots\}$.

Exercise 2.3: Find the potential energy function $V(x)$ for each of the forces in Exercise 2.2.

Solution:

We get the potential energy function $V(x)$ from formula (2.3.5) by indefinite integration.

(a)
$\frac{dV(x)}{dx} = -F(x) = -(F_0 + cx) \Rightarrow V(x) = -\int (F_0 + cx)dx = -F_0 x - \frac{1}{2}cx^2 + C$,
where C is a constant.

(b) $V(x) = -\int F_0 e^{-cx} dx = \frac{F_0}{c} e^{-cx} + C$, where C is a constant.

(c) $V(x) = -\int F_0 \cos(cx) dx = -\frac{F_0}{c} \sin(cx) + C$, where C is a constant.

Exercise 2.4: A particle of mass m is constrained to lie along a frictionless, horizontal plane subject to a force given by the expression $F(x) = -kx$. It is projected from $x = 0$ to the right along the positive x direction with initial kinetic energy $T_0 = \frac{1}{2}kA^2$. k and A are positive constants. Find (**a**) the potential energy function $V(x)$ for this force; (**b**) the kinetic energy, and (**c**) the total energy of the particle as a function of its position. (**d**) Find the turning points of the motion. (**e**) Sketch the potential, kinetic, and total energy functions. (*Optional*: Use *Mathcad* or *Mathematica* to plot these functions. Set k and A equal to 1.)

Solution:

(a) We use formula (2.3.5) for the potential energy:

$\frac{dV(x)}{dx} = -F(x) = kx \Rightarrow V(x) = \int kx dx = \frac{1}{2}kx^2 + C$, where C is a constant.

(b) We use formula (2.3.3) for the kinetic energy:

$\frac{dT(x)}{dx} = F(x) \Rightarrow T(x) = -\frac{1}{2}kx^2 + C'$, where C' is a constant.

We know that $T(x = 0) = T_0 = \frac{1}{2}kA^2 \Rightarrow C' = \frac{1}{2}kA^2$. Thus:

$T(x) = -\frac{1}{2}kx^2 + \frac{1}{2}kA^2$.

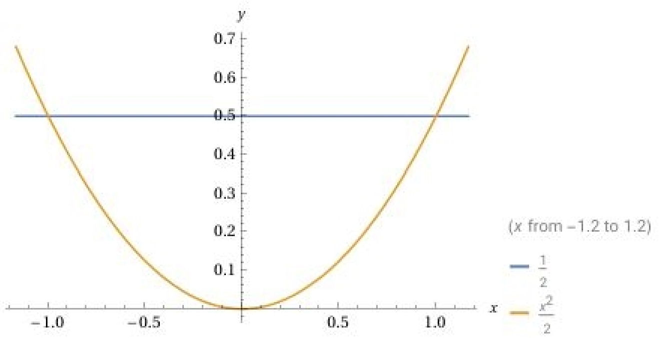
(c) We get the total energy from formula (2.3.8):

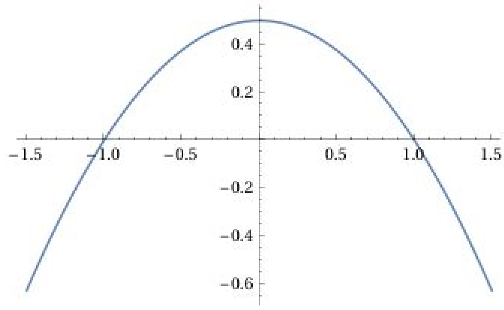
$E = E(x) = T(x) + V(x) = \frac{1}{2}k(A^2 - x^2) + \frac{1}{2}kx^2 + C \Rightarrow E = \frac{1}{2}kA^2 + C$, where C is a constant (same as for $V(x)$).

(d) For the turning points of motion we have $E = V(x)$:

$\frac{1}{2}kA^2 + C = \frac{1}{2}kx^2 + C \Rightarrow x^2 = A^2 \Rightarrow x = \pm A$.

(e) For the plots we have (plotted with online Mathematica):





Here in the left picture the yellow curve is the potential energy $V(x)$ and the blue line is the total energy E. In the right picture the curve is the kinetic energy $T(x)$.

Exercise 2.5: As in the exercise above, the particle is projected to the right with initial kinetic energy T_0, but subject to a force $F(x) = -kx + \frac{kx^3}{A^2}$, where k and A are positive constants. Find (**a**) the potential energy function $V(x)$ for this force; (**b**) the kinetic energy, and (**c**) the total energy of the particle as a function of its position. (**d**) Find the turning points of the motion and the condition the total energy of the particle must satisfy if its motion is to exhibit turning points. (**e**) Sketch the potential, kinetic, and total energy functions. (*Optional*: Use *Mathcad* or *Mathematica* to plot these functions. Set k and A equal to 1.)

Solution:

(a) We use formula (2.3.5) for the potential energy:

$\frac{dV(x)}{dx} = -F(x) = kx - \frac{kx^3}{A^2} \Rightarrow V(x) = \int (kx - \frac{kx^3}{A^2})dx = \frac{1}{2}kx^2 - \frac{kx^4}{4A^2} + C,$ where $C = V(0) = V_0$ is a constant.

(b) We use formula (2.3.3) for the kinetic energy:

$\frac{dT(x)}{dx} = F(x) \Rightarrow T(x) = -\frac{1}{2}kx^2 + \frac{kx^4}{4A^2} + C'$, where C' is a constant.

If we have that $T(x=0) = T_0 = \frac{1}{2}kA^2$, we get $C' = \frac{1}{2}kA^2$, which gives us:

$T(x) = -\frac{1}{2}kx^2 + \frac{kx^4}{4A^2} + \frac{1}{2}kA^2.$

(c) We get the total energy from formula (2.3.8):

$E = E(x) = T(x) + V(x) = \frac{1}{2}kA^2 + C = \frac{1}{2}kA^2 + V_0$, where $C = V_0$ is a constant (same as for $V(x)$).

(d) The turning points exist if $E \leq V(x)$. The inequality gives us:

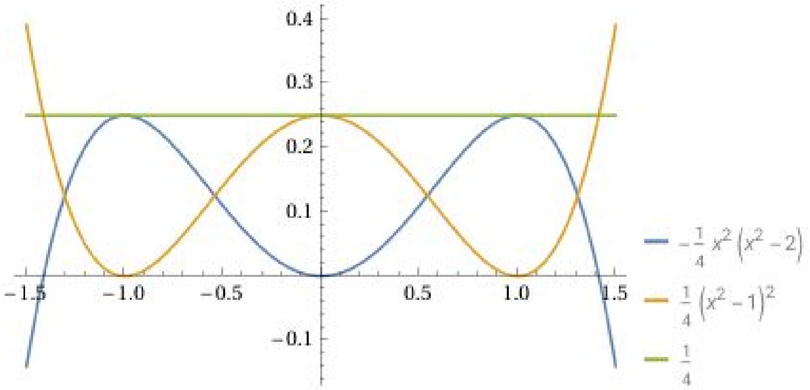
$E \leq \frac{1}{2}kx^2 - \frac{kx^4}{4A^2} \Rightarrow x^4 - 2A^2x^2 + \frac{4A^2E}{k} \leq 0.$

When is the polynomial $f(x) = x^4 - 2A^2x^2 + \frac{4A^2E}{k} = 0$? We can use the quadratic formula to solve for x:

$x^2 = \frac{-(-2a^2) \pm \sqrt{(-2a^2)^2 - 4 \cdot 1 \cdot \frac{4A^2E}{k}}}{2 \cdot 1} = A^2(1 \pm \sqrt{A^2 - \frac{4E}{k}})$

For x to be real, we have to have $A^2 - \frac{4E}{k} \geq 0 \Rightarrow E \leq \frac{1}{4}kA^2$. Thus if $E = \frac{1}{2}kA^2$ (assuming $V_0 = 0$) as in part (c), the motion will go on perpetually towards the right. But if $E = \frac{1}{4}kA^2$, then we see that we have turning points $x^2 = A^2 \Rightarrow x = \pm A$.

(e) For the plots we have (plotted with online Mathematica):



Here the blue curve is the potential energy function $V(x)$, the yellow curve is the kinetic energy $T(x)$ and the green line is the total energy E.

Exercise 2.6: A particle of mass m moves along a frictionless, horizontal plane with a speed given by $v(x) = \frac{\alpha}{x}$, where x is its distance from the origin and α is a positive constant. Find the force $F(x)$ to which the particle is subject.

Solution:

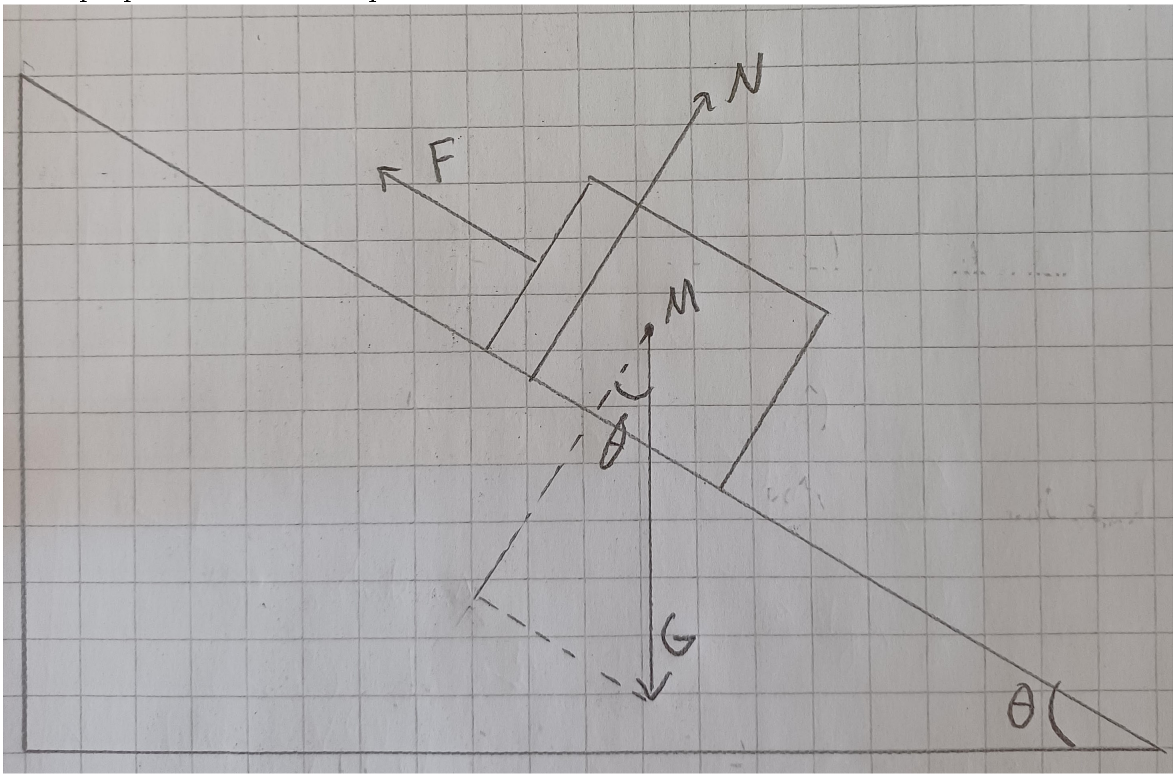
We can use formulas (2.3.1) and (2.3.2):

$$F(x) = m\ddot{x} = mv\frac{dv}{dx} = m \cdot \frac{\alpha}{x} \cdot (-\frac{\alpha}{x^2}) = -\frac{m\alpha^2}{x^3}.$$

Exercise 2.7: A block of mass M has a string of mass m attached to it. A force $\mathbf{F}$ is applied to the string, and it pulls the block up a frictionless plane that is inclined at an angle θ to the horizontal. Find the force that the string exerts on the block.

Solution:

This is not a true force diagram (because it contains 2 objects: the plane and the block) of the block of a mass M, but it does show the forces acting upon the block. $\mathbf{G}$ is gravity from the Earth, $\mathbf{N}$ is the normal force exerted on the block by the plane, which cancels the component of the gravitational force perpendicular to the plane and $\mathbf{F}$ is the force that we want to find.



From the "diagram", we see that $|\mathbf{F}| \geq Mg\sin\theta$ for the block to move upwards along the plane. The force $\mathbf{F}$ is aligned parallel to the plane.