

Analytical Mechanics

Exercises 2.8-2.14

(Exercise descriptions [with possible slight modifications] from Analytical Mechanics by Fowles and Cassiday, 7th International Student Edition. Solutions by: Waves and Tensors)

Exercise 2.8: Given that the velocity of a particle in rectilinear motion varies with the displacement x according to the equation:

$$\dot{x} = bx^{-3}$$

where b is a positive constant, find the force acting on the particle as a function of x . (*Hint:* $F = m\ddot{x} = m\dot{x}\frac{d\dot{x}}{dx}$)

Solution:

We use the hint:

$$F(x) = m\ddot{x} = m\frac{d\dot{x}}{dt} = m\frac{dx}{dt}\frac{d\dot{x}}{dx} = m\dot{x}\frac{d\dot{x}}{dx} = m \cdot \frac{b}{x^3} \cdot (-3) \cdot \frac{b}{x^4} = -3mb^2x^{-7}$$

Exercise 2.9: A baseball (radius = 0.0366 m, mass = 0.145 kg) is dropped from rest at the top of the Empire State Building (height = 1250 ft \approx 381 m). Calculate (a) the initial potential energy of the baseball, (b) its final kinetic energy, and (c) the total energy dissipated by the falling baseball by computing the line integral of the force of air resistance along the baseball's total distance of fall. Compare this last result to the difference between the baseball's initial potential energy and its final kinetic energy. (*Hint: in part (c) make approximations when evaluating the hyperbolic functions obtained in carrying out the line integral.*)

Solution:

We have $x_0 \approx 381$ m, $x_f = 0$, $D = 2 \cdot 0.0366$ m, $g = 9.8$ m/s², $m = 0.145$ kg and $v_0 = 0$.

(a) We set $V(x_f) = 0$, so:

$$-mg = -\frac{dV(x)}{dx} \Rightarrow V(x) = mgx + C \Rightarrow V(x_f) = 0 = C \Rightarrow V(x_0) = mgx_0 \approx 541.4 \text{ J}$$

(b) If the air resistance is quadratic and the baseball reaches terminal speed before hitting the ground (assumptions!), we can use formula (2.4.10):

$$v_f = v_t = \sqrt{\frac{mg}{c_2}},$$

where $c_2 = 0.22 \cdot D^2$ ($[c_2] = \frac{\text{N} \cdot \text{s}^2}{\text{m}^2}$). We get as the final kinetic energy:

$$T_f = \frac{1}{2}mv_f^2 = \frac{m^2g}{2c_2} \approx 87.4 \text{ J}$$

(c) We know that $F(v) = -c_2v^2$ and we can use formula (2.4.17) for v^2 :

$$v^2 = v_t^2(1 - e^{-\frac{2gx}{v_t^2}}) + v_0^2 e^{-\frac{2gx}{v_t^2}} = v_t^2(1 - e^{-\frac{2gx}{v_t^2}}),$$

where x now is the distance the baseball has dropped from the top of the building. The work done by the air resistance is thus:

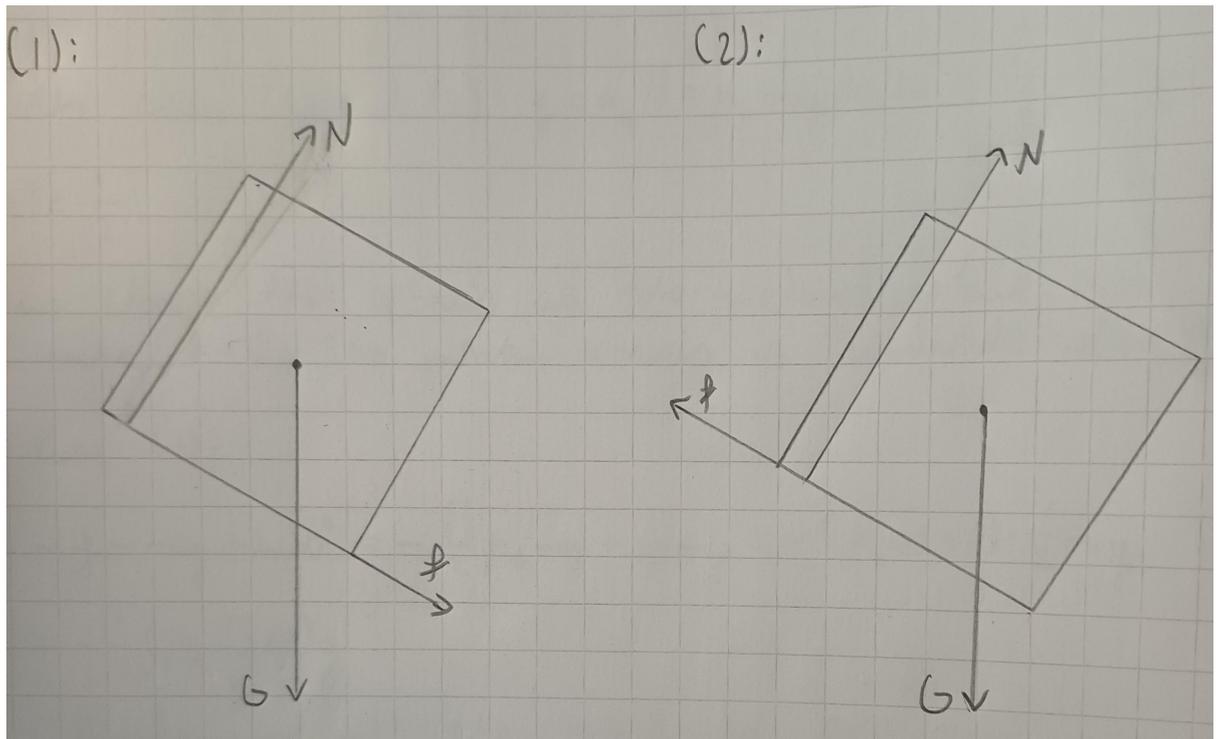
$$W = \int_0^{x_0} F(x)dx = -c_2v_t^2 \int_0^{x_0} (1 - e^{-\frac{2gx}{v_t^2}})dx = -c_2v_t^2(x_0 + \frac{v_t^2}{2g}e^{-\frac{2gx_0}{v_t^2}} - \frac{v_t^2}{2g}) \approx -454.2 \text{ J}$$

Now $V(x_0) + W \approx T_f$. W is negative, because the air resistance force is dissipative and points upwards towards the sky whereas the motion of the

baseball is down towards the earth. Reasons for why $V(x_0) + W$ is not exactly T_f might be because of calculational rounding errors, the fact that the baseball starts at rest so the air resistance is linear and not quadratic at the beginning of the motion and also because the baseball might not have reached full terminal speed upon hitting the ground as was assumed in part (b).

Exercise 2.10: A block of wood is projected up an inclined plane with initial speed v_0 . If the inclination of the plane is 30° and the coefficient of sliding friction $\mu_k = 0.1$, find the total time for the block to return to the point of projection.

Solution:



Force diagram (1) is the block going up and force diagram (2) is the block coming back down.

In both cases (1) and (2) $f = \mu_k N = \mu_k mg \cos 30^\circ = \frac{\sqrt{3}}{2} \mu_k mg$. Note that $\sin 30^\circ = \frac{1}{2}$.

Let us choose the positive x -axis as "upwards" parallel to the inclined plane and $x = 0$ when $t = 0$.

For case (1):

$$m\ddot{x} = -f - mg \sin 30^\circ = -\frac{\sqrt{3}}{2} \mu_k mg - \frac{1}{2} mg = -(\mu_k \sqrt{3} + 1) \cdot \frac{1}{2} mg \Rightarrow \ddot{x} = -\frac{1}{2} (\mu_k \sqrt{3} + 1) g$$

By indefinite integration we get the speed for case (1):

$$\dot{x} = \int -\frac{1}{2} (\mu_k \sqrt{3} + 1) g dt = -\frac{1}{2} (\mu_k \sqrt{3} + 1) gt + C, \quad (*)$$

where C is the constant of integration. We know that $\dot{x}(0) = v_0 = C$. Integrating again we get:

$$x = \int \left(-\frac{1}{2}(\mu_k\sqrt{3} + 1)gt + v_0\right) dt = -\frac{1}{4}(\mu_k\sqrt{3} + 1)gt^2 + v_0t + C',$$

where C' is the constant of integration. We know that $x(0) = 0 = C'$. At the maximum distance up the inclined plane $\dot{x}(t_{up}) = 0$ so we get from (*):

$$t_{up} = \frac{2v_0}{(\mu_k\sqrt{3} + 1)g}.$$

The maximum distance the block travels is $x_{max} = x(t_{up})$:

$$x_{max} = -\frac{1}{4}(\mu_k\sqrt{3} + 1)g \cdot \left(\frac{2v_0}{(\mu_k\sqrt{3} + 1)g}\right)^2 + \frac{2v_0^2}{(\mu_k\sqrt{3} + 1)g} = \frac{v_0^2}{(\mu_k\sqrt{3} + 1)g}.$$

For case (2):

$$m\ddot{x} = f - mg \sin 30^\circ = \frac{\sqrt{3}}{2}\mu_k mg - \frac{1}{2}mg = (\mu_k\sqrt{3} - 1) \cdot \frac{1}{2}mg \Rightarrow \ddot{x} = \frac{1}{2}(\mu_k\sqrt{3} - 1)g$$

By indefinite integration we get the speed for case (2):

$$\dot{x} = \int \frac{1}{2}(\mu_k\sqrt{3} - 1)g dt = \frac{1}{2}(\mu_k\sqrt{3} - 1)gt + A,$$

where A is the constant of integration. We know that at the top $\dot{x} = 0$. We reset the time to $t = 0$ at the top for case (2) to eliminate the integration constant and to avoid having to solve a quadratic equation with the quadratic solution formula. So now $\dot{x}(0) = 0$ which gives us $\dot{x} = 0 = A \Rightarrow \dot{x} = \frac{1}{2}(\mu_k\sqrt{3} - 1)gt$. Integrating again we get:

$$x(t) = \frac{1}{4}(\mu_k\sqrt{3} - 1)gt^2 + A',$$

where A' is an integration constant. Now in our resetted time system at $t = 0, x(0) = x_{max}$, which gives us $A' = x_{max}$. Thus for case (2) the time when $x = 0$ is t_{down} :

$$\begin{aligned} \frac{1}{4}(\mu_k\sqrt{3} - 1)gt_{down}^2 + x_{max} = 0 &\Rightarrow t_{down} = \pm \sqrt{\frac{4x_{max}}{(1 - \mu_k\sqrt{3})g}} \\ &= \sqrt{\frac{4v_0^2}{(1 + \mu_k\sqrt{3})(1 - \mu_k\sqrt{3})g^2}} \\ &= \frac{2v_0}{g} \frac{1}{\sqrt{1 - 3\mu_k^2}}. \end{aligned}$$

The total time for the block to go up and down is thus T :

$$T = t_{up} + t_{down} = \frac{2v_0}{(\mu_k\sqrt{3} + 1)g} + \frac{2v_0}{g} \frac{1}{\sqrt{1 - 3\mu_k^2}} = \frac{2v_0}{g} \left(\frac{1}{1 + \mu_k\sqrt{3}} + \frac{1}{\sqrt{1 - 3\mu_k^2}} \right)$$

Exercise 2.11: A metal block of mass m slides on a horizontal surface that has been lubricated with a heavy oil so that the block suffers a viscous resistance that varies as the $\frac{3}{2}$ power of the speed:

$$F(v) = -cv^{3/2}$$

If the initial speed of the block is v_0 at $x = 0$, show that the block cannot travel farther than $2mv_0^{1/2}/c$.

Solution:

$$\begin{aligned} F = m\ddot{x} = mv \frac{dv}{dx} = -cv^{\frac{3}{2}} &\Rightarrow \int_{v_0}^0 v^{-\frac{1}{2}} dv = -\frac{c}{m} \int_0^{x_{max}} dx \\ &\Rightarrow -2v_0^{\frac{1}{2}} = -\frac{c}{m} x_{max} \\ &\Rightarrow x_{max} = \frac{2mv_0^{\frac{1}{2}}}{c}. \end{aligned}$$

Exercise 2.12: A gun is fired straight up. Assuming that the air drag on the bullet varies quadratically with speed, show that the speed varies with height according to the equations:

$$v^2 = Ae^{-2kx} - \frac{g}{k} \quad (\text{upward motion})$$

$$v^2 = \frac{g}{k} - Be^{2kx} \quad (\text{downward motion})$$

in which A and B are constants of integration, g is the acceleration of gravity, and $k = c_2/m$, where c_2 is the drag constant and m is the mass of the bullet. (*Note:* x is measured positive upward, and the gravitational force is assumed to be constant.)

Solution:

For the upward motion we write:

$$m\ddot{x} = mv\frac{dv}{dx} = \frac{1}{2}m\frac{d(v^2)}{dx} = -mg - c_2v^2 \Rightarrow \frac{d(v^2)}{dx} = -2g\left(1 + \frac{kv^2}{g}\right)$$

We set $u = 1 + \frac{kv^2}{g} \Rightarrow \frac{du}{dx} = \frac{k}{g}\frac{d(v^2)}{dx} = -2ku$. Thus the indefinite integral becomes:

$$\int \frac{1}{u} du = \ln(u) = \int -2k dx = -2kx + C \Rightarrow u = e^C e^{-2kx}.$$

Thus:

$$1 + \frac{kv^2}{g} = e^C e^{-2kx} \Rightarrow v^2 = \frac{e^C g}{k} e^{-2kx} - \frac{g}{k} = Ae^{-2kx} - \frac{g}{k},$$

where $A = \frac{e^C g}{k}$ is a constant of integration.

For the downward motion we write:

$$m\ddot{x} = mv\frac{dv}{dx} = \frac{1}{2}m\frac{d(v^2)}{dx} = -mg + c_2v^2 \Rightarrow \frac{d(v^2)}{dx} = -2g\left(1 - \frac{kv^2}{g}\right)$$

We set $u = 1 - \frac{kv^2}{g} \Rightarrow \frac{du}{dx} = -\frac{k}{g}\frac{d(v^2)}{dx} = 2ku$. Thus the indefinite integral becomes:

$$\int \frac{1}{u} du = \ln(u) = \int 2k dx = 2kx + C' \Rightarrow u = e^{C'} e^{2kx}.$$

Thus:

$$1 - \frac{kv^2}{g} = e^{C'} e^{2kx} \Rightarrow v^2 = -\frac{e^{C'} g}{k} e^{2kx} + \frac{g}{k} = \frac{g}{k} - Be^{2kx},$$

where $B = \frac{e^{C'} g}{k}$ is a constant of integration.

Exercise 2.13: Use the above result to show that, when the bullet hits the ground on its return, the speed is equal to the expression:

$$\frac{v_0 v_t}{(v_0^2 + v_t^2)^{1/2}}$$

in which v_0 is the initial upward speed and

$$v_t = (mg/c_2)^{1/2} = \text{terminal speed} = (g/k)^{1/2}$$

(This result allows one to find the fraction of the initial kinetic energy lost through air friction.)

Solution:

For the upwards motion:

$$v_0^2 = Ae^{-2k \cdot 0} - \frac{g}{k} = A - \frac{g}{k} \Rightarrow A = \frac{g}{k} + v_0^2.$$

At the top $x = x_{max}$ and $v^2 = 0$:

$$\begin{aligned} v^2 = 0 &= \left(\frac{g}{k} + v_0^2\right)e^{-2kx_{max}} - \frac{g}{k} = (v_t^2 + v_0^2)e^{-2kx_{max}} - v_t^2 \Rightarrow e^{-2kx_{max}} = \frac{v_t^2}{v_t^2 + v_0^2} \\ &\Rightarrow e^{2kx_{max}} = \frac{v_t^2 + v_0^2}{v_t^2}. \end{aligned}$$

For the downwards motion:

$$v^2 = 0 = \frac{g}{k} - Be^{2kx_{max}} = v_t^2 - B \cdot \frac{v_t^2 + v_0^2}{v_t^2} \Rightarrow B = \frac{v_t^4}{v_0^2 + v_t^2}.$$

The final speed v_f when the bullet hits the ground $x = 0$ is then:

$$\begin{aligned} v_f^2 &= \frac{g}{k} - \frac{v_t^4}{v_0^2 + v_t^2} \cdot e^{2k \cdot 0} = v_t^2 - \frac{v_t^4}{v_0^2 + v_t^2} = \frac{v_t^4 + v_0^2 v_t^2 - v_t^4}{v_0^2 + v_t^2} = \frac{v_0^2 v_t^2}{v_0^2 + v_t^2} \\ &\Rightarrow v_f = \pm \frac{v_0 v_t}{(v_0^2 + v_t^2)^{1/2}} = \frac{v_0 v_t}{(v_0^2 + v_t^2)^{1/2}}. \end{aligned}$$

Exercise 2.14: A particle of mass m is released from rest a distance b from a fixed origin of force that attracts the particle according to the inverse square law:

$$F(x) = -kx^{-2}$$

Show that the time required for the particle to reach the origin is:

$$\pi\left(\frac{mb^3}{8k}\right)^{1/2}$$

Solution:

By formula (2.3.3) we have:

$$\frac{dT}{dx} = F(x) = -kx^{-2} \Rightarrow T(x) = kx^{-1} + C,$$

where C is an integration constant. We know that $T(b) = 0 = \frac{k}{b} + C \Rightarrow C = -\frac{k}{b} \Rightarrow T(x) = \frac{1}{2}mv^2 = \frac{k}{b}(bx^{-1} - 1)$. Thus:

$$v = \pm\left(\frac{2k}{mb}\right)^{1/2}(bx^{-1} - 1)^{1/2} = -\left(\frac{2k}{mb}\right)^{1/2}(bx^{-1} - 1)^{1/2} = \frac{dx}{dt},$$

where we take the minus part because the direction of the velocity is towards the origin. Next we take a definitive integral from point A at $x = b, t_0 = 0$ to point B at $x = 0$:

$$t - t_0 = t = - \int_b^0 \left(\frac{mb}{2k}\right)^{1/2} \cdot \frac{1}{\sqrt{\frac{b}{x} - 1}} dx = -\left(\frac{mb}{2k}\right)^{1/2} \int_b^0 \frac{1}{\sqrt{\frac{b}{x} - 1}} dx.$$

To evaluate this last integral, we look at the indefinite case first. Let us set $u = \frac{b}{x} \Rightarrow \frac{du}{dx} = -\frac{b}{x^2} = -\frac{u^2}{b} \Rightarrow \frac{dx}{du} = -\frac{b}{u^2}$. We get (from tablebooks, Mathematica, Google or AI):

$$\begin{aligned} \int \frac{1}{\sqrt{\frac{b}{x} - 1}} dx &= -b \int \frac{1}{u^2 \sqrt{u - 1}} du \\ &= -b \left(\frac{\sqrt{u - 1}}{u} + \tan^{-1}(\sqrt{u - 1}) + C' \right) \\ &= -b \left(\frac{\sqrt{\frac{b}{x} - 1}}{\frac{b}{x}} + \tan^{-1}(\sqrt{\frac{b}{x} - 1}) + C' \right). \end{aligned}$$

We investigate the limits to get the definitive case for the first term:

$$x = b \Rightarrow \frac{\sqrt{\frac{b}{x}-1}}{\frac{b}{x}} = 0$$

$$x \rightarrow 0^+ \Rightarrow \lim_{x \rightarrow 0^+} \frac{\sqrt{\frac{b}{x}-1}}{\frac{b}{x}} = \lim_{x \rightarrow 0^+} \frac{-\frac{b}{x^2}(\frac{x}{b}-1)^{-\frac{1}{2}}}{-\frac{b}{x^2}} = \lim_{x \rightarrow 0^+} \frac{1}{\sqrt{\frac{b}{x}-1}} = 0$$

For the second term:

$$x = b \Rightarrow \tan^{-1}\left(\sqrt{\frac{b}{x}-1}\right) = 0$$

$$x \rightarrow 0^+ \Rightarrow \lim_{x \rightarrow 0^+} \tan^{-1}\left(\sqrt{\frac{b}{x}-1}\right) = \frac{\pi}{2}$$

So finally:

$$t = -\left(\frac{mb}{2k}\right)^{\frac{1}{2}} \int_b^0 \frac{1}{\sqrt{\frac{b}{x}-1}} dx = -\left(\frac{mb}{2k}\right)^{\frac{1}{2}} \cdot (-b) \cdot \left(0 + \frac{\pi}{2}\right) = \pi \left(\frac{mb^3}{8k}\right)^{\frac{1}{2}}.$$